

AN ABSTRACT INVERSE PROBLEM FOR BOUNDARY TRIPLES WITH AN APPLICATION TO THE FRIEDRICHS MODEL

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ABSTRACT. We discuss the detectable subspaces of an operator. We analyse the relation between the M -function (the abstract Dirichlet to Neumann map) and the resolvent bordered by projections onto the detectable subspaces. The abstract results are explored further by an extensive study of the Friedrichs model, together with illustrative applications to the Schrödinger and Hain-Lüst-type models.

1. INTRODUCTION

In this paper we consider inverse problems in a boundary triple setting involving a formally adjoint pair of operators A and \tilde{A} , as studied in [12, 13, 14, 31, 33, 34, 35]. We define, and develop formulae for, the detectable subspace (see Definition 2.7) associated with the information available from the abstract Dirichlet to Neumann maps (Titchmarsh-Weyl functions) $M(\lambda)$. We examine the extent to which the following questions can be answered at a purely abstract level.

- (1) Is the function $M(\lambda)$ uniquely determined from a knowledge of resolvents reduced to the detectable subspace?
- (2) Can the resolvent, bordered by projections onto the detectable subspaces, be determined from $M(\lambda)$?
- (3) What can be said about the relationship between analytic continuation of $M(\lambda)$ and analytic continuation of bordered resolvents?
- (4) What is the relationship between the rank of the jump in $M(\lambda)$ and the rank of the jump in the bordered resolvent across a line of essential spectrum, w.l.o.g. the real axis, when one has a limiting absorption principle?
- (5) To what extent can the detectable subspace be explicitly described?

Illustrative examples include the Schrödinger operator and Hain-Lüst-type models which we also examined in [13]. However the main concrete example studied in this paper is the Friedrichs Model, discussed at length in Sections 7, 8, 9, together with the relevant appendices. These results reveal many connections to problems in modern complex analysis, including the theory of Hankel and Toeplitz operators, and demonstrate the interplay between complex analysis and operator theory in the description of the detectable subspace (see e.g. the appearance of the Riesz-Nevanlinna factorisation theorem in Theorem 9.3). We consider the Friedrichs model as a key example for the development of the theory of detectable subspaces, because it allows a precise description of the structure of the detectable subspace in many cases, while exhibiting such a variety of behaviours that one can hardly expect to obtain a description of the space in all cases in unique terms. It shows the problem of reconstruction of the detectable part of the operator from the M -function, well-known for Sturm-Liouville problems [10, 37], is not always possible. Our results for this example show that the detectable part of the operator can partially be recovered

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from the M -function. At least in the symmetric case we would expect this recovery to be possible up to unitary equivalence [43].

The paper is arranged as follows. Section 2 introduces boundary triples, M -functions, solution operators and the detectable subspace. Section 3 shows the concrete realizations of these abstract objects for Schrödinger operators, Hain-Lüst-type operators and the Friedrichs model. Sections 4, 5, 6 present various abstract results concerning the relationship between the bordered resolvent and the M -function. In particular, in Section 4, we prove that the M -function is uniquely determined by one bordered resolvent. It can be reconstructed from one bordered resolvent and two closed solution operator ranges or by two bordered resolvents associated with different boundary conditions. We show that the bordered resolvent can be determined from the M -function and a family of solution operator ranges. Section 5 examines simultaneous analytic continuation of the M -function and bordered resolvents, while Section 6 deals with jumps of the M -function and the bordered resolvent across the essential spectrum.

Sections 7 onwards, including the appendices, deal with the Friedrichs model. In Section 7 we consider the reconstruction of the M -function from one restricted resolvent for the Friedrichs model. Sections 8 and 9 deal with determining the detectable subspace for various combinations of the parameters of the model. In both these sections, results and techniques from complex analysis will be important; whilst in Section 8 Hankel operators will make an appearance, the results in Section 9 rely on the theory of Toeplitz operators. Many of the proofs from these sections can be found in the appendices.

We conclude this introduction by mentioning that there has been an explosion of interest in boundary triples in the last decade, in particular around their application to partial differential equations usually in the self-adjoint case (see, e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 14, 17, 18, 19, 20, 21, 22, 23, 24, 25, 30, 32, 39, 40, 43]). Some interesting ODE applications have also appeared, such as Mikhailets' and Sobolev's study [36] of the common eigenvalue problem. Generalisations to relations have been studied by Derkach, Hassi, Malamud and de Snoo [15, 16]. However the situation with inverse problems remains problematic: one of the striking differences between Schrödinger operators in dimension $d = 1$ and dimension $d > 1$ is that, for $d > 1$, the potential can be uniquely recovered from a knowledge of the Dirichlet to Neumann map at a single value of the spectral parameter [26]. The fact that this is not true for $d = 1$ [10] already indicates that abstract techniques will generally be of limited value unless supplemented by a detailed study of the concrete operators to which they will be applied.

2. DEFINITION OF THE DETECTABLE SUBSPACE AND SOME PROPERTIES

We use the following assumptions and notation throughout our article.

- (1) A, \tilde{A} are closed, densely defined operators on domains in a Hilbert space H .
- (2) A and \tilde{A} are an adjoint pair, i.e. $A^* \supseteq \tilde{A}$ and $\tilde{A}^* \supseteq A$.

Proposition 2.1. [31, (Lyantze, Storozh '83)]. *For each adjoint pair of closed densely defined operators on H , there exist “boundary spaces” \mathcal{H}, \mathcal{K} and “trace operators”*

$$\Gamma_1 : D(\tilde{A}^*) \rightarrow \mathcal{H}, \quad \Gamma_2 : D(\tilde{A}^*) \rightarrow \mathcal{K}, \quad \tilde{\Gamma}_1 : D(A^*) \rightarrow \mathcal{K} \quad \text{and} \quad \tilde{\Gamma}_2 : D(A^*) \rightarrow \mathcal{H}$$

such that for $u \in D(\tilde{A}^)$ and $v \in D(A^*)$ we have an abstract Green formula*

$$(1) \quad \left\langle \tilde{A}^* u, v \right\rangle_H - \left\langle u, A^* v \right\rangle_H = \left\langle \Gamma_1 u, \tilde{\Gamma}_2 v \right\rangle_{\mathcal{H}} - \left\langle \Gamma_2 u, \tilde{\Gamma}_1 v \right\rangle_{\mathcal{K}}.$$

The trace operators $\Gamma_1, \Gamma_2, \tilde{\Gamma}_1$ and $\tilde{\Gamma}_2$ are bounded with respect to the graph norm. The pair (Γ_1, Γ_2) is surjective onto $\mathcal{H} \times \mathcal{K}$ and $(\tilde{\Gamma}_1, \tilde{\Gamma}_2)$ is surjective onto $\mathcal{K} \times \mathcal{H}$. Moreover, we have

$$(2) \quad D(A) = D(\tilde{A}^*) \cap \ker \Gamma_1 \cap \ker \Gamma_2 \quad \text{and} \quad D(\tilde{A}) = D(A^*) \cap \ker \tilde{\Gamma}_1 \cap \ker \tilde{\Gamma}_2.$$

The collection $\{\mathcal{H} \oplus \mathcal{K}, (\Gamma_1, \Gamma_2), (\tilde{\Gamma}_1, \tilde{\Gamma}_2)\}$ is called a boundary triple for the adjoint pair A, \tilde{A} .

Malamud and Mogilevskii [35] use this setting to define Weyl M -functions associated with boundary triples. In [14], we used a slightly different setting in which the boundary conditions and Weyl function contain an additional operator $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. We now summarize some results from [14] for the convenience of the reader.

Definition 2.2. Let $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ and $\tilde{B} \in \mathcal{L}(\mathcal{H}, \mathcal{K})$. We define extensions of A and \tilde{A} (respectively) by

$$A_B := \tilde{A}^*|_{\ker(\Gamma_1 - B\Gamma_2)} \text{ and } \tilde{A}_{\tilde{B}} := A^*|_{\ker(\tilde{\Gamma}_1 - \tilde{B}\tilde{\Gamma}_2)}.$$

In the following, we assume $\rho(A_B) \neq \emptyset$, in particular A_B is a closed operator. For $\lambda \in \rho(A_B)$, we define the M -function via

$$M_B(\lambda) : \text{Ran}(\Gamma_1 - B\Gamma_2) \rightarrow \mathcal{K}, \quad M_B(\lambda)(\Gamma_1 - B\Gamma_2)u = \Gamma_2 u \text{ for all } u \in \ker(\tilde{A}^* - \lambda)$$

and for $\lambda \in \rho(\tilde{A}_{\tilde{B}})$, we define

$$\tilde{M}_{\tilde{B}}(\lambda) : \text{Ran}(\tilde{\Gamma}_1 - \tilde{B}\tilde{\Gamma}_2) \rightarrow \mathcal{H}, \quad \tilde{M}_{\tilde{B}}(\lambda)(\tilde{\Gamma}_1 - \tilde{B}\tilde{\Gamma}_2)v = \tilde{\Gamma}_2 v \text{ for all } v \in \ker(A^* - \lambda).$$

It will follow from Lemma 2.5 that $M_B(\lambda)$ and $\tilde{M}_{\tilde{B}}(\lambda)$ are well defined for $\lambda \in \rho(A_B)$ and $\lambda \in \rho(\tilde{A}_{\tilde{B}})$, respectively. Moreover, in our situation $\text{Ran}(\Gamma_1 - B\Gamma_2) = \mathcal{H}$ and $\text{Ran}(\tilde{\Gamma}_1 - \tilde{B}\tilde{\Gamma}_2) = \mathcal{K}$, so the M -functions are defined on the whole spaces.

Definition 2.3. (Solution Operator) For $\lambda \in \rho(A_B)$, we define the linear operator $S_{\lambda, B} : \text{Ran}(\Gamma_1 - B\Gamma_2) \rightarrow \ker(\tilde{A}^* - \lambda)$ by

$$(3) \quad (\tilde{A}^* - \lambda)S_{\lambda, B}f = 0, \quad (\Gamma_1 - B\Gamma_2)S_{\lambda, B}f = f,$$

i.e. $S_{\lambda, B} = \left((\Gamma_1 - B\Gamma_2)|_{\ker(\tilde{A}^* - \lambda)} \right)^{-1}$. For $\lambda \in \rho(\tilde{A}_{\tilde{B}})$, we define the linear operator $\tilde{S}_{\lambda, B^*} : \text{Ran}(\tilde{\Gamma}_1 - \tilde{B}^*\tilde{\Gamma}_2) \rightarrow \ker(A^* - \lambda)$ by

$$(4) \quad (A^* - \lambda)\tilde{S}_{\lambda, B^*}f = 0, \quad (\tilde{\Gamma}_1 - \tilde{B}^*\tilde{\Gamma}_2)\tilde{S}_{\lambda, B^*}f = f.$$

All following results have a corresponding version for the quantities $\tilde{M}_{\tilde{B}}$, \tilde{S}_{λ, B^*} etc. obtained from the formally adjoint problem.

Remark 2.4. (1) As we are not interested in characterising all closed extensions of A , in this paper we will assume for simplicity that $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. A discussion of all closed extensions of A in the boundary triple setting can be found in [12].

(2) Note that $M_B(\lambda) = \Gamma_2 S_{\lambda, B}$.

(3) M -functions associated with different boundary conditions are related by the Aronszajn-Donoghue formula (cf. also 13)

$$(5) \quad M_B(\lambda) = (I + M_B(\lambda)(B - C))M_C(\lambda) = M_C(\lambda)(I + (B - C)M_B(\lambda)).$$

The following lemma contains the results of [13, Lemma 2.4 and Corollary 2.5].

Lemma 2.5. (1) $S_{\lambda, B}$ is well-defined for $\lambda \in \rho(A_B)$.

(2) For each $f \in \text{Ran}(\Gamma_1 - B\Gamma_2)$ the map from $\rho(A_B) \rightarrow \mathcal{H}$ given by $\lambda \mapsto S_{\lambda, B}f$ is analytic.

(3) For $\lambda, \lambda_0 \in \rho(A_B)$ we have

$$(6) \quad S_{\lambda, B} = S_{\lambda_0, B} + (\lambda - \lambda_0)(A_B - \lambda)^{-1}S_{\lambda_0, B}.$$

The difference of two resolvents of the operator can be related to the M -function by Krein-type resolvent formulae, such as

$$(7) \quad \begin{aligned} (A_C - \lambda)^{-1} - (A_B - \lambda)^{-1} &= S_{\lambda,C}(I + (B - C)M_B(\lambda))(\Gamma_1 - B\Gamma_2)(A_C - \lambda)^{-1} \\ &= S_{\lambda,C}(I + (B - C)M_B(\lambda))(C - B)\Gamma_2(A_C - \lambda)^{-1}, \end{aligned}$$

for $B, C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ and $\lambda \in \rho(A_B) \cap \rho(A_C)$ (see [13, Theorem 2.6]).

The following formula already appears in some proofs in [13, 14], but due to its importance in our later analysis, we state and prove it here.

Lemma 2.6. *For every $F \in D(\tilde{A}^*)$ and $v \in D(A^*)$ and $\lambda \in \rho(A_B)$ we have*

$$(8) \quad \left\langle F - (A_B - \lambda)^{-1}(\tilde{A}^* - \lambda)F, (A^* - \bar{\lambda}I)v \right\rangle = \left\langle M_B(\lambda)f, (\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)v \right\rangle_{\mathcal{K}} - \left\langle f, \tilde{\Gamma}_2v \right\rangle_{\mathcal{H}}$$

where $f = (\Gamma_1 - B\Gamma_2)F$.

Proof. Set $w := F - (A_B - \lambda)^{-1}(\tilde{A}^* - \lambda)F$. Then $w \in \ker(\tilde{A}^* - \lambda)$, so

$$M_B(\lambda)(\Gamma_1 - B\Gamma_2)w = M_B(\lambda)f = \Gamma_2w \text{ and } \Gamma_1w = (\Gamma_1 - B\Gamma_2 + B\Gamma_2)w = (I + BM_B(\lambda))f.$$

Green's identity (1) for any $v \in D(A^*)$ gives

$$\begin{aligned} -\left\langle w, (A^* - \bar{\lambda})v \right\rangle_H &= \left\langle (\tilde{A}^* - \lambda)w, v \right\rangle_H - \left\langle w, (A^* - \bar{\lambda})v \right\rangle_H \\ &= \left\langle \Gamma_1w, \tilde{\Gamma}_2v \right\rangle_{\mathcal{H}} - \left\langle \Gamma_2w, \tilde{\Gamma}_1v \right\rangle_{\mathcal{K}} \\ &= \left\langle (I + BM_B(\lambda))f, \tilde{\Gamma}_2v \right\rangle_{\mathcal{H}} - \left\langle M_B(\lambda)f, \tilde{\Gamma}_1v \right\rangle_{\mathcal{K}} \\ &= \left\langle f, \tilde{\Gamma}_2v \right\rangle_{\mathcal{H}} - \left\langle M_B(\lambda)f, (\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)v \right\rangle_{\mathcal{K}}. \end{aligned}$$

□

We are now ready to define one of the main concepts of the paper, the detectable subspaces, introduced in [13].

Definition 2.7. *Fix $\mu_0 \notin \sigma(A_B)$. We define the spaces*

$$(9) \quad \mathcal{S}_B = \text{Span}_{\delta \notin \sigma(A_B)} (A_B - \delta I)^{-1} \text{Ran}(S_{\mu_0, B}),$$

$$(10) \quad \mathcal{T}_B = \text{Span}_{\mu \notin \sigma(A_B)} \text{Ran}(S_{\mu, B}),$$

and similarly,

$$(11) \quad \tilde{\mathcal{S}}_{B^*} = \text{Span}_{\delta \notin \sigma(\tilde{A}_{B^*})} (\tilde{A}_{B^*} - \delta I)^{-1} \text{Ran}(\tilde{S}_{\tilde{\mu}, B^*}),$$

$$(12) \quad \tilde{\mathcal{T}}_{B^*} = \text{Span}_{\mu \notin \sigma(\tilde{A}_{B^*})} \text{Ran}(\tilde{S}_{\mu, B^*}).$$

We call $\overline{\mathcal{S}}_B$ and $\overline{\tilde{\mathcal{S}}_{B^*}}$ the detectable subspaces.

We now consider the dependence of these spaces on μ_0 and B .

Proposition 2.8. (1) *Let $B \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. Assume that there is a sequence $(z_n)_{n \in \mathbb{N}}$ in \mathbb{C} with $|z_n| \rightarrow \infty$ and $(\|z_n(A_B - z_n I)^{-1}\|)_{n \in \mathbb{N}}$ is bounded. Then we have $\overline{\mathcal{S}}_B = \overline{\mathcal{T}}_B$. In particular, $\overline{\mathcal{S}}_B$ is independent of μ_0 .*

(2) *Let $B, C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$. If $\rho(A_B) \cup \rho(A_C) \subseteq \overline{\rho(A_B) \cap \rho(A_C)}$, then $\overline{\mathcal{T}}_B = \overline{\mathcal{T}}_C$.*

(3) *Suppose that for all $B, C \in \mathcal{L}(\mathcal{K}, \mathcal{H})$, we have $\rho(A_B) \cup \rho(A_C) \subseteq \overline{\rho(A_B) \cap \rho(A_C)}$. Then $\overline{\mathcal{T}}_B = \overline{\text{Span}_{\lambda \in \Lambda} \ker(\tilde{A}^* - \lambda)}$, where $\Lambda = \bigcup_{C \in \mathcal{L}(\mathcal{K}, \mathcal{H})} \rho(A_C)$.*

Proof. (1) This is shown in [13, Lemma 3.1].

(2) From [14, Proposition 4.5] we have

$$(13) \quad S_{\lambda,C}(I - (C - B)\Gamma_2 S_{\lambda,B}) = S_{\lambda,B},$$

we note that $\text{Ran}(S_{\lambda,B}) = \text{Ran}(S_{\lambda,C})$ whenever $\lambda \in \rho(A_B) \cap \rho(A_C)$. Now assume $\lambda \in \rho(A_B) \cap \sigma(A_C)$. We need to show that $\text{Ran}(S_{\lambda,B}) \subseteq \overline{\mathcal{T}'}$ where

$$\mathcal{T}' = \text{Span}_{\mu \in \rho(A_B) \cap \rho(A_C)} \text{Ran}(S_{\mu,B}).$$

By assumption, there exists a sequence $(\lambda_n)_{n \in \mathbb{N}}$ in $\rho(A_B) \cap \rho(A_C)$ with $\lambda_n \rightarrow \lambda$. Let $u = S_{\lambda,B}f$. We have

$$S_{\lambda_n,B} - S_{\lambda,B} = (\lambda_n - \lambda)(A_B - \lambda_n)^{-1}S_{\lambda,B}.$$

Therefore,

$$\|S_{\lambda_n,B}f - S_{\lambda,B}f\| \leq |\lambda_n - \lambda| \|(A_B - \lambda_n)^{-1}\| \|S_{\lambda,B}f\|.$$

As $n \rightarrow \infty$, $\|(A_B - \lambda_n)^{-1}\| \rightarrow \|(A_B - \lambda)^{-1}\| < \infty$, so $S_{\lambda_n,B}f \rightarrow S_{\lambda,B}f$ which completes the proof.

(3) This follows immediately from the previous part of the proposition. \square

Remark 2.9. We note that the conditions in parts 1 and 3 of the proposition are satisfied in many interesting cases, in particular in the case of ‘weak’ perturbations of selfadjoint operators.

Throughout the remainder of this article, we will assume that the spaces $\overline{\mathcal{S}_B}$ and $\overline{\mathcal{T}_B}$ coincide, are independent of B and equal $\text{Span}_{\lambda \in \Lambda} \ker(\tilde{A}^* - \lambda)$. To avoid cumbersome notation, we shall denote all these spaces by $\overline{\mathcal{S}}$. We shall also denote \mathcal{S}_B by \mathcal{S} and \mathcal{T}_B by \mathcal{T} when no confusion can arise. We will generally refer to $\overline{\mathcal{S}}$ as the detectable subspace.

In [13, Lemma 3.4], it is shown that $\overline{\mathcal{S}}$ is a regular invariant space of the resolvent of the operator A_B : that is, $(A_B - \mu I)^{-1}\overline{\mathcal{S}} = \overline{\mathcal{S}}$ for all $\mu \in \rho(A_B)$.

From (10) and Proposition 2.8, part 3, we get

$$(14) \quad \mathcal{S}^\perp = \bigcap_{B, \lambda \in \rho(A_B)} \ker(S_{\lambda,B}^*).$$

Moreover, from [14, Proposition 3.9] we have

$$\ker(S_{\lambda,B}^*) = \ker\left(\tilde{\Gamma}_2(\tilde{A}_{B^*} - \bar{\lambda})^{-1}\right).$$

We now assume that $h \in \mathcal{S}^\perp$. Then we have $\tilde{\Gamma}_2(\tilde{A}_{B^*} - \bar{\lambda})^{-1}h = 0$ for all suitable B and λ . Fixing B and λ and setting

$$(15) \quad y_B = (\tilde{A}_{B^*} - \bar{\lambda})^{-1}h,$$

we get $\tilde{\Gamma}_2 y_B = 0$ and hence $\tilde{\Gamma}_1 y_B = B^* \tilde{\Gamma}_2 y_B = 0$, so y_B satisfies any homogeneous boundary condition and lies in the domain of the minimal operator.

Hence,

$$(16) \quad \mathcal{S}^\perp = \{h \in H : \forall B^*, \lambda \in \rho(\tilde{A}_{B^*}), \quad \tilde{\Gamma}_i(\tilde{A}_{B^*} - \lambda)^{-1}h = 0, \text{ for } i = 1, 2\}.$$

Remark 2.10. Determining the detectable subspace \mathcal{S} is closely related to the problem of observability in systems theory. Indeed, from (14) and (16) the space \mathcal{S}^\perp (at least formally) coincides with the ‘non-observable for all time’ subspace $N(\Theta)$ of a system $\Theta = (A, B, C, D)$ (see [45]), in which $A = \tilde{A}_{B^*}$, $B = \tilde{\mathcal{S}}_{\lambda_0, B^*}$ for some $\lambda_0 \in \rho(\tilde{A}_{B^*})$, $C = \tilde{\Gamma}_2(\tilde{A}^* - \lambda_0)$, $D = 0$, though there are several differences between the notions:

- (1) *The corresponding system can be highly awkward to construct and requires involving unbounded operators.*
- (2) *In systems theory the subspace of un-observable states is generated by the resolvent in one half plane only (corresponding to positive times t only). In our construction, the spectral parameter runs through the whole resolvent set. It is well known that when the resolvent set consists of several unconnected domains, developing the linear set by the resolvent essentially depends on the choice of the component.*
- (3) *We do not require the operator \tilde{A}_{B^*} to be the generator of a semigroup. In particular, the resolvent set may have a complicated geometrical structure. If \tilde{A}_{B^*} is a generator, the resolvent in (16) can be replaced by the positive and negative time semigroups.*

Despite these differences, the similarity to the observability problem may be fruitful for analysing detectability, both in general and in particular examples. For more connections between boundary triples and systems theory, we refer to [44].

3. EXAMPLE OPERATORS

In this section we shall examine three different concrete operators which will be used in the following to illustrate the power and also the limitations of the theory. For the first of these we show that $\bar{\mathcal{S}}$ is the whole underlying Hilbert space; for the second example we refer the reader to some previous work, where we show that $\bar{\mathcal{S}}$ may or may not be the whole space; for the third example, we calculate the function $M_B(\lambda)$, in preparation for the substantial work in Sections 8, 9 and the Appendix, which shows that the characterization of $\bar{\mathcal{S}}$ may be very subtle for this seemingly innocuous model.

3.1. Schrödinger problems. For complex valued $q \in L^\infty(0, 1)$, consider

$$(17) \quad Lu = \left(-\frac{d^2}{dx^2} + q \right) u \text{ and } \tilde{L}u = \left(-\frac{d^2}{dx^2} + \bar{q} \right) u \quad \text{on } [0, 1].$$

Let $Au = Lu$ and $\tilde{A}u = \tilde{L}u$ with $D(A) = D(\tilde{A}) = H_0^2(0, 1)$. Then $\tilde{A}^*u = Lu$ and $A^*u = \tilde{L}u$ with $D(\tilde{A}^*) = D(A^*) = H^2(0, 1)$ and for $u, v \in H^2(0, 1)$

$$\langle \tilde{A}^*u, v \rangle - \langle u, A^*v \rangle = \langle \Gamma_1 u, \Gamma_2 v \rangle - \langle \Gamma_2 u, \Gamma_1 v \rangle,$$

where

$$\Gamma_1 u = \begin{pmatrix} -u'(1) \\ u'(0) \end{pmatrix}, \quad \Gamma_2 u = \begin{pmatrix} u(1) \\ u(0) \end{pmatrix}.$$

In particular, $\Gamma_1 = \tilde{\Gamma}_1$, $\Gamma_2 = \tilde{\Gamma}_2$ and $\mathcal{H} = \mathcal{K} = \mathbb{C}^2$.

Let $\theta(x, \lambda)$ and $\phi(x, \lambda)$ be solutions of $\tilde{L}u = \bar{\lambda}u$ which satisfy $\theta(0, \lambda) = 0$, $\theta'(0, \lambda) = 1$ and $\phi(0, \lambda) = 1$, $\phi'(0, \lambda) = 0$. Let y_B be as in (15). Then by the variation of constants formula, there exist C, \tilde{C} such that

$$y_B(x, \lambda) = \int_0^x \phi(t, \lambda) h(t) dt \theta(x, \lambda) + \int_x^1 \theta(t, \lambda) h(t) dt \phi(x, \lambda) + C\theta(x, \lambda) + \tilde{C}\phi(x, \lambda).$$

y_B satisfies $\Gamma_1 y_B = 0 = \Gamma_2 y_B$. We choose λ so that it is not a Dirichlet eigenvalue. Then

$$\begin{aligned} y_B(0, \lambda) &= \int_0^1 \theta h dt + \tilde{C} = 0, & y_B(1, \lambda) &= \left(\int_0^1 \phi h dt + C \right) \theta(1, \lambda) + \tilde{C}\phi(1, \lambda) = 0, \\ y'_B(0, \lambda) &= C = 0, & y'_B(1, \lambda) &= \left(\int_0^1 \phi h dt + C \right) \theta'(1, \lambda) + \tilde{C}\phi'(1, \lambda) = 0. \end{aligned}$$

This simplifies to

$$\begin{aligned} \int_0^1 \phi h dt \theta(1, \lambda) - \int_0^1 \theta h dt \phi(1, \lambda) &= 0, \\ \int_0^1 \phi h dt \theta'(1, \lambda) - \int_0^1 \theta h dt \phi'(1, \lambda) &= 0. \end{aligned}$$

As the Wronskian of θ and ϕ is non-zero, we have

$$\int_0^1 \theta h dt = \int_0^1 \phi h dt = 0.$$

This holds for almost all λ . Analyticity in λ implies that these equations hold for all λ . Choosing λ to run through the Dirichlet eigenvalues shows that \bar{h} is orthogonal to all Dirichlet eigenfunctions and also to any possible root vectors. Hence, $h \equiv 0$ and we have proved the following result which is consistent with the Borg Uniqueness Theorem [10, 37].

Proposition 3.1. *For the Schrödinger operator we have $\bar{\mathcal{S}} = L^2(0, 1)$.*

3.2. Hain-Lüst-type operators. Let

$$(18) \quad \tilde{A}^* = \begin{pmatrix} -\frac{d^2}{dx^2} + q(x) & \tilde{w}(x) \\ w(x) & u(x) \end{pmatrix} \quad \text{and} \quad A^* = \begin{pmatrix} -\frac{d^2}{dx^2} + \overline{q(x)} & \overline{w(x)} \\ \overline{\tilde{w}(x)} & \overline{u(x)} \end{pmatrix},$$

where q , u , \tilde{w} and w are L^∞ -functions, and the domain of the operators is given by

$$(19) \quad D(\tilde{A}^*) = D(A^*) = H^2(0, 1) \times L^2(0, 1).$$

It is then easy to see that

$$(20) \quad \begin{aligned} &\left\langle \tilde{A}^* \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} y \\ z \end{pmatrix}, A^* \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle \\ &= \left\langle \Gamma_1 \begin{pmatrix} y \\ z \end{pmatrix}, \Gamma_2 \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle - \left\langle \Gamma_2 \begin{pmatrix} y \\ z \end{pmatrix}, \Gamma_1 \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle, \end{aligned}$$

where

$$\mathcal{H} = \mathcal{K} = \mathbb{C}^2, \quad \Gamma_1 \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -y'(1) \\ y'(0) \end{pmatrix}, \quad \Gamma_2 \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y(1) \\ y(0) \end{pmatrix}.$$

Some information on \mathcal{S} for these operators is available in [13]. In particular we show there that if $w = \tilde{w}$ then $\bar{\mathcal{S}} \subseteq L^2(0, 1) \oplus L^2((w^{-1}(\{0\})^c)$ (where Ω^c denotes the complement of a set Ω) and so if w vanishes on a set of positive measure then $\bar{\mathcal{S}}$ is not the whole underlying space.

3.3. The Friedrichs model. We consider in $L^2(\mathbb{R})$ the operator A with domain

$$(21) \quad D(A) = \left\{ f \in L^2(\mathbb{R}) \mid xf(x) \in L^2(\mathbb{R}), \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx \text{ exists and is zero} \right\},$$

given by the expression

$$(22) \quad (Af)(x) = xf(x) + \langle f, \phi \rangle \psi(x),$$

where ϕ, ψ are in $L^2(\mathbb{R})$. Observe that since the constant function $\mathbf{1}$ does not lie in $L^2(\mathbb{R})$ the domain of A is dense in $L^2(\mathbb{R})$.

We first collect some results from [13] where more details and proofs can be found:

Lemma 3.2. *The adjoint of A is given on the domain*

$$(23) \quad D(A^*) = \{ f \in L^2(\mathbb{R}) \mid \exists c_f \in \mathbb{C} : xf(x) - c_f \mathbf{1} \in L^2(\mathbb{R}) \},$$

by the formula

$$(24) \quad A^* f = xf(x) - c_f \mathbf{1} + \langle f, \psi \rangle \phi.$$

Since $c_f = \lim_{R \rightarrow \infty} (2R)^{-1} \int_{-R}^R x f(x) dx$ is uniquely determined, we can define trace operators Γ_1 and Γ_2 on $D(A^*)$ as follows:

$$(25) \quad \Gamma_1 u = \lim_{R \rightarrow \infty} \int_{-R}^R u(x) dx, \quad \Gamma_2 u = c_u.$$

Note that $\Gamma_1 u = \int_{\mathbb{R}} (u(x) - c_u \mathbf{1} \text{sign}(x)(x^2 + 1)^{-1/2}) dx$, which is the expression used in [13].

Lemma 3.3. *The operators Γ_1 and Γ_2 are bounded relative to A^* and the following ‘Green’s identity’ holds:*

$$(26) \quad \langle A^* f, g \rangle - \langle f, A^* g \rangle = \Gamma_1 f \overline{\Gamma_2 g} - \Gamma_2 f \overline{\Gamma_1 g} + \langle f, \psi \rangle \langle \phi, g \rangle - \langle f, \phi \rangle \langle \psi, g \rangle.$$

We introduce an operator \tilde{A} in which the roles of ϕ and ψ are swapped:

$$(27) \quad D(\tilde{A}) = \left\{ f \in L^2(\mathbb{R}) \mid x f(x) \in L^2(\mathbb{R}), \quad \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx = 0 \right\},$$

$$(28) \quad (\tilde{A} f)(x) = x f(x) + \langle f, \psi \rangle \phi(x).$$

In view of Lemma 3.2 we immediately see that $D(\tilde{A}^*) = D(A^*)$ and that

$$(29) \quad (\tilde{A}^* f)(x) = x f(x) - c_f \mathbf{1} + \langle f, \phi \rangle \psi(x).$$

Thus \tilde{A}^* is an extension of A , A^* is an extension of \tilde{A} , and the following result is easily proved.

Lemma 3.4.

$$(30) \quad A = \tilde{A}^* \Big|_{\ker(\Gamma_1) \cap \ker(\Gamma_2)}; \quad \tilde{A} = A^* \Big|_{\ker(\Gamma_1) \cap \ker(\Gamma_2)};$$

moreover, the Green’s formula (26) can be modified to

$$(31) \quad \langle A^* f, g \rangle - \langle f, \tilde{A}^* g \rangle = \Gamma_1 f \overline{\Gamma_2 g} - \Gamma_2 f \overline{\Gamma_1 g}.$$

We finish our review from [13] with the M -function and the resolvent:

Lemma 3.5. *Suppose that $\Im \lambda \neq 0$. Then $f \in \ker(\tilde{A}^* - \lambda)$ if*

$$(32) \quad f(x) = \Gamma_2 f \left[\frac{1}{x - \lambda} - \frac{\langle (t - \lambda)^{-1}, \phi \rangle \psi(x)}{D(\lambda)} \frac{1}{x - \lambda} \right].$$

Here D is the function

$$(33) \quad D(\lambda) = 1 + \int_{\mathbb{R}} \frac{1}{x - \lambda} \psi(x) \overline{\phi(x)} dx.$$

Moreover the Titchmarsh-Weyl coefficient $M_B(\lambda)$ is given by

$$(34) \quad M_B(\lambda) = \left[\text{sign}(\Im \lambda) \pi i - \frac{\langle (t - \lambda)^{-1}, \bar{\psi} \rangle \langle (t - \lambda)^{-1}, \phi \rangle}{D(\lambda)} - B \right]^{-1}.$$

For the resolvent, we have that $(A_B - \lambda)f = g$ if and only if

$$(35) \quad f(x) = \frac{g(x)}{x - \lambda} - \frac{1}{D(\lambda)} \frac{\psi(x)}{x - \lambda} \left\langle \frac{g}{t - \lambda}, \phi \right\rangle + c_f \left[\frac{1}{x - \lambda} - \frac{1}{D(\lambda)} \frac{\psi(x)}{x - \lambda} \left\langle \frac{1}{t - \lambda}, \phi \right\rangle \right],$$

in which the coefficient c_f is given by

$$(36) \quad c_f = M_B(\lambda) \left[- \left\langle \frac{1}{t - \lambda}, \bar{g} \right\rangle + \frac{1}{D(\lambda)} \left\langle \frac{g}{t - \lambda}, \phi \right\rangle \left\langle \frac{1}{t - \lambda}, \bar{\psi} \right\rangle \right].$$

These calculations will be needed in Sections 7, 8 and the Appendix.

Remark 3.6. *There is another approach to the Friedrichs model via the Fourier transform which may appear much more natural. It is easy to check that, denoting the Fourier transform by \mathcal{F} and $\mathcal{F}f = \hat{f}$, we get*

$$\mathcal{F}A\mathcal{F}^* = i\frac{d}{dx} + \left\langle \cdot, \hat{\phi} \right\rangle \hat{\psi}, \quad D(\mathcal{F}A\mathcal{F}^*) = \{u \in H^1(\mathbb{R}) : u(0) = 0\},$$

$$\mathcal{F}\tilde{A}^*\mathcal{F}^* = i\frac{d}{dx} + \left\langle \cdot, \hat{\phi} \right\rangle \hat{\psi}, \quad D(\mathcal{F}\tilde{A}^*\mathcal{F}^*) = \{u \in L^2(\mathbb{R}) : u|_{\mathbb{R}^\pm} \in H^1(\mathbb{R}^\pm)\},$$

and

$$\mathcal{F}A_B\mathcal{F}^* = i\frac{d}{dx} + \left\langle \cdot, \hat{\phi} \right\rangle \hat{\psi},$$

$$D(\mathcal{F}A_B\mathcal{F}^*) = \left\{ u \in L^2(\mathbb{R}) : u|_{\mathbb{R}^\pm} \in H^1(\mathbb{R}^\pm), u(0^+) = \frac{B - i\pi}{B + i\pi} u(0^-) \right\},$$

where $u(0^\pm)$ denotes the limit of u at zero from the left and right, respectively. Moreover, $\Gamma_1 f = \sqrt{\pi/2}(\hat{f}(0^+) + \hat{f}(0^-))$ and $\Gamma_2 f = i(2\pi)^{-1}(\hat{f}(0^+) - \hat{f}(0^-))$. There are similar expressions for the adjoint operators and traces. In terms of extension theory it is much easier to use this Fourier representation. However, for our later calculations, the original model is more useful, as it facilitates the calculation of residues.

4. RELATION BETWEEN M -FUNCTION AND RESOLVENT ON $\bar{\mathcal{S}}$

Having introduced some concrete examples in the previous sections, we now turn our attention to what can be shown in the general setting. Our aim is to study the relation between the function M_B and the bordered resolvent $P_{\bar{\mathcal{S}}}(A_B - \lambda)^{-1}|_{\bar{\mathcal{S}}}$ where for any subspace M , P_M denotes the orthogonal projection onto M .

4.1. Information on the M -function contained in the resolvent. We first look at gaining information on the M -function from knowledge of the resolvent.

Theorem 4.1. *Let $\lambda \in \rho(A_B)$. Then $P_{\bar{\mathcal{S}}}(A_B - \lambda)^{-1}|_{\bar{\mathcal{S}}}$ uniquely determines $M_B(\lambda)$.*

In particular, if also $\lambda \in \rho(A_C)$, then $P_{\bar{\mathcal{S}}}(A_B - \lambda)^{-1}|_{\bar{\mathcal{S}}} = P_{\bar{\mathcal{S}}}(A_C - \lambda)^{-1}|_{\bar{\mathcal{S}}}$ implies that $M_B(\lambda) = M_C(\lambda)$, and, if additionally $\lambda \in \rho(A_\infty)$, then $B = C$. Here, $A_\infty = \tilde{A}^|_{\ker \Gamma_2}$.*

Proof. Assume $\widehat{M}_B(\lambda)$ is a different M -function for the same problem. By surjectivity of the trace operators there exist $F \in D(\tilde{A}^*)$ and $v \in D(A^*)$ such that

$$\left\langle M_B(\lambda)(\Gamma_1 - B\Gamma_2)F, (\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)v \right\rangle_{\mathcal{K}} \neq \left\langle \widehat{M}_B(\lambda)(\Gamma_1 - B\Gamma_2)F, (\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)v \right\rangle_{\mathcal{K}}.$$

Setting $h = S_{\mu,B}(\Gamma_1 - B\Gamma_2)F \in \mathcal{S}$ and $\tilde{h} = \tilde{S}_{\tilde{\mu},B^*}(\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)v \in \tilde{\mathcal{S}}$ and using (8), we find that

$$\left\langle (I - (A_B - \lambda)^{-1}(\mu - \lambda))h, (\tilde{\mu} - \bar{\lambda})\tilde{h} \right\rangle$$

has two different values. Therefore, $\left\langle (A_B - \lambda)^{-1}h, \tilde{h} \right\rangle$ has two different values yielding a contradiction.

If $P_{\bar{\mathcal{S}}}(A_B - \lambda)^{-1}|_{\bar{\mathcal{S}}} = P_{\bar{\mathcal{S}}}(A_C - \lambda)^{-1}|_{\bar{\mathcal{S}}}$, then by the argument above

$$\left\langle M_B(\lambda)(\Gamma_1 - B\Gamma_2)F, (\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)v \right\rangle_{\mathcal{K}} = \left\langle M_C(\lambda)(\Gamma_1 - B\Gamma_2)F, (\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)v \right\rangle_{\mathcal{K}}.$$

Choosing F and v such that $\Gamma_2 F = \tilde{\Gamma}_2 v = 0$ and $\Gamma_1 F$ and $\tilde{\Gamma}_1 v$ are arbitrary, we obtain

$$\left\langle M_B(\lambda)\Gamma_1 F, \tilde{\Gamma}_1 v \right\rangle_{\mathcal{K}} = \left\langle M_C(\lambda)\Gamma_1 F, \tilde{\Gamma}_1 v \right\rangle_{\mathcal{K}}.$$

Hence, $M_B(\lambda) = M_C(\lambda)$. If $\lambda \in \rho(A_\infty)$, then $\text{Ran } M_B(\lambda) = \mathcal{K}$ and $\ker M_C(\lambda) = \{0\}$, so from (5) we get $B = C$. \square

Note that from the knowledge of the resolvent and the range of one solution operator we can explicitly reconstruct $\overline{\mathcal{S}}$. With some extra knowledge of the problem we can reconstruct $M_B(\lambda)$ from knowledge of the bordered resolvent on $\overline{\mathcal{S}}$.

Theorem 4.2. *Assume we know $\overline{\mathcal{S}}$, $P_{\overline{\mathcal{S}}}(A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}}$ and $\overline{\text{Ran}(S_{\mu,B})}$, $\overline{\text{Ran}(\tilde{S}_{\tilde{\mu},B^*})}$ for some $(\mu, \tilde{\mu})$ with $\mu, \tilde{\mu} \in \rho(A_B)$. Then we can reconstruct $M_B(\lambda)$ uniquely if B is known.*

Proof. For $h \in \overline{\text{Ran}(S_{\mu,B})}$, $\tilde{h} \in \overline{\text{Ran}(\tilde{S}_{\tilde{\mu},B^*})}$, consider

$$H(h, \tilde{h}) = \left\langle (I - (A_B - \lambda)^{-1}(\mu - \lambda))h, (\tilde{\mu} - \bar{\lambda})\tilde{h} \right\rangle.$$

By assumption, we know $H(h, \tilde{h})$. Varying h throughout $\overline{\text{Ran}(S_{\mu,B})}$, we have that $(\Gamma_1 - B\Gamma_2)h$ runs through the whole of \mathcal{H} ; varying \tilde{h} throughout $\overline{\text{Ran}(\tilde{S}_{\tilde{\mu},B^*})}$, the values $(\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)\tilde{h}$ run through the whole of \mathcal{K} and $\tilde{\Gamma}_2\tilde{h}$ through the whole of \mathcal{H} . Using Lemma 2.6 we have for a dense set of h, \tilde{h} that

$$H(h, \tilde{h}) = \left\langle M_B(\lambda)(\Gamma_1 - B\Gamma_2)h, (\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)\tilde{h} \right\rangle_{\mathcal{K}} - \left\langle (\Gamma_1 - B\Gamma_2)h, \tilde{\Gamma}_2\tilde{h} \right\rangle_{\mathcal{H}},$$

which allows reconstruction of the M -function. \square

We look into the question of how strong the condition of knowledge of the closed ranges of solution operators needed in the theorem is. We first look at the Friedrichs model.

Proposition 4.3. *Assume we know $\text{Ran } S_{\lambda,B}$ and $\text{Ran } \tilde{S}_{\mu,B^*}$ for some λ, μ for the Friedrichs model. Moreover, assume $\text{Ran } S_{\lambda,B} \neq \text{Span}\{\frac{1}{x-\lambda}\}$ and $\text{Ran } \tilde{S}_{\mu,B^*} \neq \text{Span}\{\frac{1}{x-\mu}\}$ (which is true for generic ϕ, ψ). Then the operator is uniquely determined.*

Proof. Assume we know $\text{Ran } S_{\lambda,B}$ and $\text{Ran } \tilde{S}_{\mu,B^*}$ for some λ, μ . Choosing the elements u and v in the ranges with $c_u = 1$ and $c_v = 1$ we have from (32) that

$$u(x) = \frac{1}{x-\lambda} - \frac{\langle (t-\lambda)^{-1}, \phi \rangle}{D(\lambda)} \cdot \frac{\psi(x)}{x-\lambda} \quad \text{and} \quad v(x) = \frac{1}{x-\mu} - \frac{\langle (t-\mu)^{-1}, \psi \rangle}{\overline{D(\mu)}} \cdot \frac{\phi(x)}{x-\mu}.$$

As one of the functions ϕ, ψ is only determined up to a scalar factor, we may normalize ϕ such that

$$\frac{\langle (t-\lambda)^{-1}, \phi \rangle}{D(\lambda)} = 1.$$

This allows us to determine ψ from the first expression, which also gives us $\langle (t-\mu)^{-1}, \psi \rangle$, so $\phi(x)/\overline{D(\mu)}$ is known. Solving $v(x)$ for $\phi(x)$, we get

$$(37) \quad \phi(x) = (1 - (x-\mu)v(x)) \frac{\overline{D(\mu)}}{\langle (t-\mu)^{-1}, \psi \rangle}.$$

Multiplying by $\overline{(x-\lambda)^{-1}D(\lambda)^{-1}}$ and integrating over x , we can determine $D(\mu)D(\lambda)^{-1}$ from our normalisation of ϕ .

Inserting our expression (37) into $D(\lambda) = 1 + \langle (x-\lambda)^{-1}, \overline{\psi\phi} \rangle$, we get a second equation relating $D(\lambda)$ and $D(\mu)$, allowing us to determine $D(\mu)$ and hence $\phi(x)$. \square

Remark 4.4. *Note that, if $\text{Ran } S_{\lambda,B} = \text{Span}\{\frac{1}{x-\lambda}\}$ or $\text{Ran } \tilde{S}_{\mu,B^*} = \text{Span}\{\frac{1}{x-\mu}\}$, it is clear from (32) and (34) that the M -function in general does not contain sufficient information to recover ϕ and ψ .*

The following result shows that nothing like the result of Proposition 4.3 holds for Hain-Lüst operators and therefore no similar result can hold in the abstract setting.

Proposition 4.5. *Assume we know $\text{Ran } S_{\lambda,B}$ and $\text{Ran } \tilde{S}_{\mu,B^*}$ for some λ, μ for the Hain-Lüst operator*

$$\begin{pmatrix} -\frac{d^2}{dx^2} + q(x) & w(x) \\ \tilde{w}(x) & u(x) \end{pmatrix}.$$

Then the operator is not uniquely determined.

Proof. It is sufficient to show the claim for the case when the coefficients of the operator are real. Knowing the ranges of the solution operators corresponds to knowing kernels of the maximal operators and hence two linearly independent solutions each to both of the following equations:

$$(38) \quad \begin{pmatrix} -\frac{d^2}{dx^2} + q(x) & w(x) \\ \tilde{w}(x) & u(x) \end{pmatrix} \begin{pmatrix} y_1 \\ z_1 \end{pmatrix} = \lambda \begin{pmatrix} y_1 \\ z_1 \end{pmatrix}$$

and

$$(39) \quad \begin{pmatrix} -\frac{d^2}{dx^2} + q(x) & \tilde{w}(x) \\ w(x) & u(x) \end{pmatrix} \begin{pmatrix} y_2 \\ z_2 \end{pmatrix} = \mu \begin{pmatrix} y_2 \\ z_2 \end{pmatrix}.$$

Write the pairs of solutions as (y_1, z_1) , (\hat{y}_1, \hat{z}_1) , and (y_2, z_2) , (\hat{y}_2, \hat{z}_2) , respectively. Since

$$z_1 = \frac{\tilde{w}y_1}{u - \lambda}, \quad \hat{z}_1 = \frac{\tilde{w}\hat{y}_1}{u - \lambda}, \quad z_2 = \frac{wy_2}{u - \mu}, \quad \hat{z}_2 = \frac{w\hat{y}_2}{u - \mu},$$

setting $\alpha = \frac{\tilde{w}}{u - \lambda}$ and $\beta = \frac{w}{u - \mu}$, this can be written in the form

$$(40) \quad \begin{pmatrix} y_1 & \alpha y_1 & 0 & 0 \\ \hat{y}_1 & \alpha \hat{y}_1 & 0 & 0 \\ y_2 & 0 & \beta y_2 & 0 \\ \hat{y}_2 & 0 & \beta \hat{y}_2 & 0 \\ 0 & 0 & y_1 & \alpha y_1 \\ 0 & 0 & \hat{y}_1 & \alpha \hat{y}_1 \\ 0 & y_2 & 0 & \beta y_2 \\ 0 & \hat{y}_2 & 0 & \beta \hat{y}_2 \end{pmatrix} \begin{pmatrix} q \\ w \\ \tilde{w} \\ u \end{pmatrix} = \begin{pmatrix} \lambda y_1 + y_1'' \\ \lambda \hat{y}_1 + \hat{y}_1'' \\ \mu y_2 + y_2'' \\ \mu \hat{y}_2 + \hat{y}_2'' \\ \lambda z_1 \\ \lambda \hat{z}_1 \\ \mu z_2 \\ \mu \hat{z}_2 \end{pmatrix}.$$

A calculation shows that the matrix on the left hand side of the equation does not have full rank for any α and β , so the system is not uniquely solvable. \square

4.2. Reconstruction from two bordered resolvents. Although we are primarily concerned with inverse problems, it is still interesting to consider some forward problems with partial data arising from the restriction of the resolvent operators to the detectable subspace.

Theorem 4.6. *Assume $P_{\bar{\mathcal{S}}}(A_B - \lambda)^{-1}|_{\bar{\mathcal{S}}}$ and $P_{\bar{\mathcal{S}}}(A_C - \lambda)^{-1}|_{\bar{\mathcal{S}}}$ are known. In addition, assume that*

- (i) $\Gamma_2(A_C - \lambda)^{-1}\bar{\mathcal{S}}$ and $\tilde{\Gamma}_2(A_C - \lambda)^{-*}\bar{\bar{\mathcal{S}}}$ are known,
- (ii) $\Gamma_2(A_C - \lambda)^{-1}\bar{\mathcal{S}}$ is dense in \mathcal{H} and $\tilde{\Gamma}_2(A_C - \lambda)^{-*}\bar{\bar{\mathcal{S}}}$ is dense in \mathcal{K} ,
- (iii) $\text{Ran } (B - C)$ is dense in \mathcal{H} and $\ker (B - C) = \{0\}$. Then $M_B(\lambda)$ can be recovered.

Proof. Let $\lambda \in \rho(A_B) \cap \rho(A_C)$. Then the Krein formula (7) gives

$$(A_B - \lambda)^{-1} - (A_C - \lambda)^{-1} = S_{\lambda,C}(I + (B - C)M_B(\lambda))(B - C)\Gamma_2(A_C - \lambda)^{-1}.$$

Now let $f \in \bar{\mathcal{S}}$ and $g \in \bar{\mathcal{S}}$. Then we know

$$\langle f, (A_B - \lambda)^{-1}g \rangle - \langle f, (A_C - \lambda)^{-1}g \rangle.$$

Using (7), we obtain

$$\begin{aligned}
& \langle f, (A_B - \lambda)^{-1}g \rangle - \langle f, (A_C - \lambda)^{-1}g \rangle \\
&= \langle f, S_{\lambda,C}(I + (B - C)M_B(\lambda))(B - C)\Gamma_2(A_C - \lambda)^{-1}g \rangle \\
&= \left\langle \tilde{\Gamma}_2(A_C - \lambda)^{-*}f, (I + (B - C)M_B(\lambda))(B - C)\Gamma_2(A_C - \lambda)^{-1}g \right\rangle \\
&= \left\langle \tilde{\Gamma}_2(A_C - \lambda)^{-*}f, (B - C)\Gamma_2(A_C - \lambda)^{-1}g \right\rangle + \\
&\quad \left\langle (B - C)^*\tilde{\Gamma}_2(A_C - \lambda)^{-*}f, M_B(\lambda)(B - C)\Gamma_2(A_C - \lambda)^{-1}g \right\rangle.
\end{aligned}$$

Our assumptions now allow us to recover $M_B(\lambda)$. \square

Remark 4.7. *Alternatively, knowing the projection to $\widetilde{\mathcal{S}}$ of the derivative of the resolvent w.r.t. the boundary condition and one resolvent restricted to $\overline{\mathcal{S}}$ will suffice, as, for $C = B + \varepsilon D$ we have from (7)*

$$\frac{(A_B - \lambda)^{-1} - (A_C - \lambda)^{-1}}{\varepsilon} \rightarrow S_{\lambda,B}D\Gamma_2(A_B - \lambda)^{-1} \quad \text{as } \varepsilon \rightarrow 0.$$

If we now have the assumption of density of $D\Gamma_2(A_B - \lambda)^{-1}\overline{\mathcal{S}}$, then for $f \in \overline{\mathcal{S}}$, $g \in \widetilde{\mathcal{S}}$, since

$$\left\langle S_{\lambda,B}D\Gamma_2(A_B - \lambda)^{-1}f, g \right\rangle = \left\langle D\Gamma_2(A_B - \lambda)^{-1}f, \tilde{\Gamma}_2(A_B - \lambda)^{-*}g \right\rangle,$$

and $\langle S_{\lambda,B}D\Gamma_2(A_B - \lambda)^{-1}f, g \rangle$ is known from the derivative, while $D\Gamma_2(A_B - \lambda)^{-1}f$ is known from the restricted resolvent, knowing the projection of the derivative to $\widetilde{\mathcal{S}}$ corresponds to knowing $\tilde{\Gamma}_2(A_B - \lambda)^{-*}|_{\widetilde{\mathcal{S}}}$.

4.3. Information on the resolvent from the M -function. The following result gives some insight to the inverse problem of reconstructing A_B from $M_B(\lambda)$. From examples later, we will see that knowledge of the M -function does not allow reconstruction of the bordered resolvent (see Remark 8.17).

Theorem 4.8. *Assume we know $M_B(\lambda)$ for all $\lambda \in \rho(A_B)$, $\text{Ran}(S_{\mu,B})$ for all $\mu \in \Lambda$ and $\text{Ran}(\tilde{S}_{\tilde{\mu},B^*})$ for all $\tilde{\mu} \in \tilde{\Lambda}$, where $\Lambda \subseteq \rho(A_B)$ and $\tilde{\Lambda} \subseteq \rho(A_B^*)$ are dense subsets. Then we can reconstruct $P_{\overline{\mathcal{S}}}(A_B - \lambda)^{-1}P_{\overline{\mathcal{S}}}$ for all $\lambda \in \rho(A_B)$.*

Proof. Let $\mu \in \Lambda$ and $\tilde{\mu} \in \tilde{\Lambda}$. Consider (8) for any $F \in \text{Ran}(S_{\mu,B})$ and $v \in \text{Ran}(\tilde{S}_{\tilde{\mu},B^*})$. Then

$$\begin{aligned}
\left\langle F - (A_B - \lambda)^{-1}(\tilde{A}^* - \lambda)F, (A^* - \bar{\lambda}I)v \right\rangle &= \left\langle F - (A_B - \lambda)^{-1}(\mu - \lambda)F, (\tilde{\mu} - \bar{\lambda}I)v \right\rangle \\
&= -\left\langle (\Gamma_1 - B\Gamma_2)F, \tilde{\Gamma}_2v \right\rangle_{\mathcal{H}} + \left\langle M_B(\lambda)(\Gamma_1 - B\Gamma_2)F, (\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)v \right\rangle_{\mathcal{K}}.
\end{aligned}$$

We know the r.h.s. of this equation for any $F \in \text{Ran}(S_{\mu,B})$, $v \in \text{Ran}(\tilde{S}_{\tilde{\mu},B^*})$ and $\lambda \in \rho(A_B)$, so we know

$$\left\langle F - (A_B - \lambda)^{-1}(\mu - \lambda)F, (\tilde{\mu} - \bar{\lambda}I)v \right\rangle.$$

Choosing $\lambda \neq \mu$ and $\lambda \neq \tilde{\mu}$, we know

$$\left\langle (A_B - \lambda)^{-1}F, v \right\rangle = \left\langle P_{\overline{\mathcal{S}}}(A_B - \lambda)^{-1}P_{\overline{\mathcal{S}}}F, v \right\rangle$$

for any $F \in \text{Ran}(S_{\mu,B})$, $v \in \text{Ran}(\tilde{S}_{\tilde{\mu},B^*})$ and $\lambda \in \rho(A_B) \setminus (\{\mu\} \cup \{\tilde{\mu}\})$. By continuity, we know it for all $\lambda \in \rho(A_B)$.

Since $\text{Span}\{\text{Ran}(S_{\mu,B}) : \mu \in \Lambda\}$ is dense in $\overline{\mathcal{S}}$ and $\text{Span}\{\text{Ran}(\tilde{S}_{\tilde{\mu},B^*}) : \tilde{\mu} \in \tilde{\Lambda}\}$ is dense in $\overline{\tilde{\mathcal{S}}}$, using boundedness of $P_{\overline{\mathcal{S}}}(A_B - \lambda)^{-1}P_{\overline{\mathcal{S}}}$ gives the result. \square

5. ANALYTIC CONTINUATION

In preparation for the discussion in Section 6 of jumps in $M_B(\lambda)$ and of the bordered resolvent across the essential spectrum, we now discuss the relationship between the analytic continuation of $M_B(\lambda)$ and analytic continuation of the bordered resolvent of A_B , both initially defined on the resolvent set of A_B .

Theorem 5.1. *Let $\mu, \tilde{\mu} \in \rho(A_B)$. Assume that for any $F \in \text{Ran}(S_{\mu,B})$, $v \in \text{Ran}(\tilde{S}_{\tilde{\mu},B^*})$,*

$$\langle (A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}} F, v \rangle$$

admits an analytic continuation to some region D of the complex plane (possibly on a different Riemann sheet). Then $M_B(\cdot)$ admits an analytic continuation to the same region D .

Proof. Given $f \in \mathcal{H}$ and $\tilde{f} \in \mathcal{K}$, choose $F = S_{\mu,B}f$ and $v = \tilde{S}_{\tilde{\mu},B^*}\tilde{f}$. Then (8) becomes

$$\langle F - (\mu - \lambda)(A_B - \lambda)^{-1}F, (\tilde{\mu} - \bar{\lambda})v \rangle = -\langle f, \tilde{\Gamma}_2 v \rangle_{\mathcal{H}} + \langle M_B(\lambda)f, \tilde{f} \rangle_{\mathcal{K}}$$

and the l.h.s. admits analytic continuation, so the r.h.s. does as well. \square

Lemma 5.2. *For $\mu \in \rho(A_B)$,*

$$\left(\frac{d}{d\lambda} S_{\cdot,B} \right) (\mu) = (A_B - \mu)^{-1} S_{\mu,B}.$$

Proof. From (6), we have

$$\frac{S_{\lambda,B}f - S_{\mu,B}f}{\lambda - \mu} = (A_B - \lambda)^{-1} S_{\mu,B}f,$$

which immediately proves the result. \square

Theorem 5.3. *Assume $M_B(\cdot)$ admits an analytic continuation to some region D of the complex plane (possibly on a different Riemann sheet). Let $\mu, \tilde{\mu} \in \rho(A_B)$. Then*

$$\langle (A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}} F, v \rangle$$

admits an analytic continuation to the same region D for any $F \in \text{Ran}(S_{\mu,B})$, $v \in \text{Ran}(\tilde{S}_{\tilde{\mu},B^})$, apart from possible simple poles at μ and $\tilde{\mu}$. If $\mu = \tilde{\mu}$, a pole of order 2 is possible at this point.*

Proof. Let $F \in \text{Ran}(S_{\mu,B})$, $v \in \text{Ran}(\tilde{S}_{\tilde{\mu},B^*})$. By assumption the r.h.s. of (8) admits analytic continuation, hence so does the l.h.s., given by

$$\langle F - (\mu - \lambda)(A_B - \lambda)^{-1}F, (\tilde{\mu} - \bar{\lambda})v \rangle.$$

Since $\langle F, (\tilde{\mu} - \bar{\lambda})v \rangle$ is clearly analytic, we have that

$$(\mu - \lambda)(\tilde{\mu} - \bar{\lambda}) \langle (A_B - \lambda)^{-1}F, v \rangle$$

is analytic which gives the desired result. \square

Remark 5.4. *We can extend the set of those vectors for which $\langle (A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}} F, v \rangle$ admits analytic continuation by developing vectors on both sides by taking linear combinations and using the resolvents of $\overline{A_B}$ and \tilde{A}_{B^*} respectively. However, we should not expect the result to extend to the whole of $\overline{\mathcal{S}}$ (or $\tilde{\mathcal{S}}$) and therefore the bordered resolvent will not necessarily admit analytic continuation.*

It is interesting to note that poles of $\langle (A_B - \lambda)^{-1}|_{\overline{S}}F, v \rangle$ at μ and $\tilde{\mu}$ do arise in concrete examples, though they may sometimes be cancelled by other terms.

Example 5.5. Let $\mu \in \mathbb{C}^-$, $\tilde{\mu} \in \mathbb{C}^+$. Consider an example of the Friedrichs model from Section 3.3, where $\phi \in H_2^-$, $\psi \in H_2^+$ are rational functions with poles in suitable half-planes such that $\psi(\lambda)\overline{\phi(\bar{\lambda})}$ does not have poles at μ or $\tilde{\mu}$. Then

$$F_\mu := \frac{1}{x - \mu} \in H_2^+ \cap \ker(\tilde{A}^* - \mu) \quad \text{and} \quad v_{\tilde{\mu}} := \frac{1}{x - \tilde{\mu}} \in H_2^- \cap \ker(A^* - \tilde{\mu}).$$

We consider the analytic continuation of the functions $M_B(\cdot)$ and $\langle (A_B - \cdot)^{-1}F_\mu, v_{\tilde{\mu}} \rangle$ from the upper to the lower half-plane.

From (8), we get for $\lambda \in \mathbb{C}^+$

$$(41) \quad (\lambda - \mu)(\lambda - \tilde{\mu}) \langle (A_B - \lambda)^{-1}F_\mu, v_{\tilde{\mu}} \rangle = - \left\langle M_B(\lambda)(\Gamma_1 - B\Gamma_2)F_\mu, (\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)v_{\tilde{\mu}} \right\rangle \\ + \left\langle (\Gamma_1 - B\Gamma_2)F_\mu, \tilde{\Gamma}_2 v_{\tilde{\mu}} \right\rangle + \langle F_\mu, (\tilde{\mu} - \bar{\lambda})v_{\tilde{\mu}} \rangle.$$

Now,

$$\langle F_\mu, v_{\tilde{\mu}} \rangle = \left\langle \frac{1}{x - \mu}, \frac{1}{x - \tilde{\mu}} \right\rangle = 0, \\ (\Gamma_1 - B\Gamma_2)F_\mu = (\Gamma_1 - B\Gamma_2)\frac{1}{x - \mu} = -\pi i - B \\ (\tilde{\Gamma}_1 - B^*\tilde{\Gamma}_2)v_{\tilde{\mu}} = \pi i - B^* \quad \text{and} \quad \tilde{\Gamma}_2 v_{\tilde{\mu}} = 1.$$

Thus (41) gives

$$(\lambda - \mu)(\lambda - \tilde{\mu}) \langle (A_B - \lambda)^{-1}F_\mu, v_{\tilde{\mu}} \rangle = -M_B(\lambda)(-\pi i - B)(-\pi i - B) + (-\pi i - B) \\ = -(\pi i + B)(M_B(\lambda)(\pi i + B) + 1),$$

or

$$(42) \quad \langle (A_B - \lambda)^{-1}F_\mu, v_{\tilde{\mu}} \rangle = -\frac{(\pi i + B)(M_B(\lambda)(\pi i + B) + 1)}{(\lambda - \mu)(\lambda - \tilde{\mu})} \\ = \frac{-2\pi i \left[\frac{\pi i - B}{\pi i + B} - 2\pi i \psi(\lambda)\overline{\phi(\bar{\lambda})} \right]^{-1}}{(\lambda - \mu)(\lambda - \tilde{\mu})}.$$

From (34) we get

$$(43) \quad M_B(\lambda) = \left[\pi i + \frac{4\pi^2 \psi(\lambda)\overline{\phi(\bar{\lambda})}}{1 + 2\pi i \psi(\lambda)\overline{\phi(\bar{\lambda})}} - B \right]^{-1} = \left[-\pi i - B + \frac{2\pi i}{1 + 2\pi i \psi(\lambda)\overline{\phi(\bar{\lambda})}} \right]^{-1}.$$

This $M_B(\cdot)$ admits an analytic continuation to the lower half-plane while the analytic continuation of $\langle (A_B - \cdot)^{-1}F_\mu, v_{\tilde{\mu}} \rangle$ given by (42) has poles at μ and $\tilde{\mu}$.

In the case when $B \neq -\pi i$, to cancel the poles in (42) we need to chose poles of the analytic continuation of $\psi(\lambda)\overline{\phi(\bar{\lambda})}$ to lie at μ and $\tilde{\mu}$. Note that the poles appearing in Theorem 5.3 should not be confused with resonances (poles of the analytic continuation of M_B). Here the resonances are due to zeroes of $\frac{\pi i - B}{\pi i + B} - 2\pi i \psi(\lambda)\overline{\phi(\bar{\lambda})}$ in formula (42).

6. ABSTRACT THEORY: RELATION BETWEEN JUMPS OF M_B AND BORDERED RESOLVENT

We consider the case in which A_B and A_B^* have essential spectrum lying on the real axis; we wish to examine in what sense $M_B(\lambda)$ jumps across the real axis, and how this may be related to a jump in the resolvent $(A_B - \lambda)^{-1}$.

Assumption 1. *We assume that there exist countable families $\{f_i\}_{i \in I}$, $\{w_j\}_{j \in \bar{I}}$ in \mathcal{H} and \mathcal{K} respectively, whose closed linear spans are \mathcal{H} and \mathcal{K} , and that for these families the inner products $\langle M_B(\lambda)f_i, w_j \rangle$ lie in both the Nevanlinna classes $N(\mathbb{C}_\pm)$. This implies that they have non-tangential boundary values $\langle M_B(k \pm i0)f_i, w_j \rangle$ for a.e. $k \in \mathbb{R}$. The class $N(\mathbb{C}_\pm)$ consists of all meromorphic functions on \mathbb{C}_\pm which can be represented as the quotient of two bounded analytic functions in the corresponding half-plane (see [29]).*

Our first result is that this assumption is equivalent to an assumption on the resolvent.

Lemma 6.1. *The functions $\langle M_B(\lambda)f_i, w_j \rangle$ lie in $N(\mathbb{C}_\pm)$ if and only if, for every $\mu \notin \sigma(A_B)$ and $\tilde{\mu} \notin \sigma(\tilde{A}_{B^*})$, the functions $\langle (A_B - \lambda)^{-1}S_{\mu,B}f_i, \tilde{S}_{\tilde{\mu},B^*}w_j \rangle$ lie in $N(\mathbb{C}_\pm)$.*

Proof. Starting with the fundamental identity

$$\langle (\tilde{A}^* - \lambda)u, v \rangle - \langle u, (A^* - \bar{\lambda})v \rangle = \langle \Gamma_1 u, \tilde{\Gamma}_2 v \rangle - \langle \Gamma_2 u, \tilde{\Gamma}_1 v \rangle$$

and making the choices $u = (A_B - \lambda)^{-1}S_{\mu,B}f_i$, $v = \tilde{S}_{\tilde{\mu},B^*}w_j$ leads to

$$(44) \quad \langle S_{\mu,B}f_i, w_j \rangle - \langle (A_B - \lambda)^{-1}S_{\mu,B}f_i, (\tilde{\mu} - \bar{\lambda})\tilde{S}_{\tilde{\mu},B^*}w_j \rangle = \\ \langle \Gamma_2(A_B - \lambda)^{-1}S_{\mu,B}f_i, \tilde{\Gamma}_1\tilde{S}_{\tilde{\mu},B^*}w_j \rangle - \langle \Gamma_2(A_B - \lambda)^{-1}S_{\mu,B}f_i, \tilde{\Gamma}_1\tilde{S}_{\tilde{\mu},B^*}w_j \rangle.$$

If the functions $\langle M_B(\lambda)f_i, w_j \rangle$ lie in $N(\mathbb{C}_\pm)$ then, thanks to the identity

$$(45) \quad M_B(\lambda) = \Gamma_2(I + (\lambda - \mu)(A_B - \lambda)^{-1})S_{\mu,B},$$

the terms $\langle \Gamma_2(A_B - \lambda)^{-1}S_{\mu,B}f_i, \cdot \rangle$ appearing in (44) also lie in $N(\mathbb{C}_\pm)$, so

$$\langle (A_B - \lambda)^{-1}S_{\mu,B}f_i, (\tilde{\mu} - \bar{\lambda})\tilde{S}_{\tilde{\mu},B^*}w_j \rangle$$

lies in $N(\mathbb{C}_\pm)$. This implies that $\langle (A_B - \lambda)^{-1}S_{\mu,B}f_i, \tilde{S}_{\tilde{\mu},B^*}w_j \rangle$ lies in $N(\mathbb{C}_\pm)$.

The converse result is immediate from equation (45): if inner products of the form

$$\langle (A_B - \lambda)^{-1}S_{\mu,B}f_i, \tilde{S}_{\tilde{\mu},B^*}w_j \rangle$$

lie in $N(\mathbb{C}_\pm)$ then so do the inner products $\langle M_B(\lambda)f_i, w_j \rangle$. \square

Theorem 6.2. *Suppose that Assumption 1 holds and that $\varepsilon|\langle M_B(k \pm i\varepsilon)f, w \rangle| \rightarrow 0$ as $\varepsilon \searrow 0$, for a.e. $k \in \mathbb{R}$ for f, w in a dense countable subset of the boundary spaces. Choose sets $\{\mu_i\}_{i \in I}$, $\{\tilde{\mu}_j\}_{j \in \bar{I}}$ non-real and outside $\sigma(A_B)$. Let $F_i = S_{\mu_i,B}f_i$, $v_j = \tilde{S}_{\tilde{\mu}_j,B^*}w_j$. Then for a.e. $k \in \mathbb{R}$,*

$$\text{rank} \left(\left[P_{\{v_j\}}(A_B - \lambda)^{-1}P_{\{F_i\}} \right]_{\lambda=k} \right) = \text{rank} \left(\left[P_{\{w_j\}}M_B(\lambda)P_{\{f_i\}} \right]_{\lambda=k} \right),$$

where by $P_{\{v_j\}}$ and $P_{\{w_j\}}$ we denote the projections onto the indicated one-dimensional spaces, and $[\cdot]_{\lambda=k}$ denotes the jump between $\lambda = k + i\varepsilon$ and $\lambda = k - i\varepsilon$ as $\varepsilon \searrow 0$.

In order to prove Theorem 6.2 we require the following lemma.

Lemma 6.3. *The collections $\{F_i\}_{i \in I}$ and $\{v_j\}_{j \in \bar{I}}$ are both linearly independent.*

Proof. We give the proof for the collection $\{F_i\}_{i \in I}$; the remaining case is similar. Assume that there are some constants α_i such that $\sum_i \alpha_i F_i = 0$. Let $\zeta \in \mathbb{C}$: applying $(\tilde{A}^* - \zeta)^k$ for some $k \in \mathbb{N}$ we get $\sum_i \alpha_i (\mu_i - \zeta)^k F_i = 0$. First, assume all the μ_i are distinct. Then we can choose i_0 and ζ such that $|\mu_{i_0} - \zeta| > |\mu_i - \zeta|$ for $i \neq i_0$. Letting $k \rightarrow \infty$ we deduce that $\alpha_{i_0} F_{i_0} = 0$. Proceeding in this way we get $\alpha_i = 0$ for all i as long as the μ_i are distinct.

If we have a collection of μ_i which are all equal, say for $i \in J$, where J is some index set, then we can prove that for some appropriately chosen ζ we have $0 = \sum_{i \in J} \alpha_i (\mu_i - \zeta)^k S_{\mu_i, B} f_i$ giving, for $\ell \in J$, $S_{\mu_\ell, B} \sum_{i \in J} \alpha_i f_i = 0$. This implies that $\sum_{i \in J} \alpha_i f_i = 0$ and hence that $\alpha_i = 0$ for all $i \in J$, by linear independence of the $\{f_i\}$. \square

Corollary 6.4. *The collections $\{(\mu_i - k)F_i\}$ and $\{(\bar{\mu}_j - k)v_j\}$ are both linearly independent as long as $\mu_i \neq k$, $\bar{\mu}_j \neq k$, for all i, j .*

Proof of Theorem 6.2. We use the fundamental identity (8), which yields

$$(46) \quad \left\langle F_i - (A_B - \lambda)^{-1}(\tilde{A}^* - \lambda)F_i, (A^* - \bar{\lambda}I)v_j \right\rangle = - \left\langle f_i, \tilde{\Gamma}_2 v_j \right\rangle_{\mathcal{H}} + \left\langle M_B(\lambda)f_i, w_j \right\rangle_{\mathcal{K}}$$

for all $i \in I$, $j \in \tilde{I}$ and $\lambda = k + i\varepsilon$. The jump at k of the right hand side is clearly given by $[\langle M_B(k)f_i, w_j \rangle_{\mathcal{K}}]$, which for convenience we denote by $\langle [M_B](k)f_i, w_j \rangle_{\mathcal{K}}$. By our assumptions on the $\{f_i\}$ and $\{w_j\}$, this is nonzero if and only if $[M_B](k) \neq 0$.

Now consider the left hand side of (46). Clearly, $\left\langle F_i, (A^* - \bar{\lambda}I)v_j \right\rangle$ has no jump.

Defining

$$(47) \quad G(\lambda) := \left\langle (A_B - \lambda)^{-1}F_i, A^*v_j \right\rangle + \left\langle (A_B - \lambda)^{-1}\tilde{A}^*F_i, v_j \right\rangle$$

the negative of the remaining term on the left hand side of (46) is

$$(48) \quad \begin{aligned} \left\langle (A_B - \lambda)^{-1}(\tilde{A}^* - \lambda)F_i, (A^* - \bar{\lambda})v_j \right\rangle &= \left\langle (A_B - \lambda)^{-1}\tilde{A}^*F_i, A^*v_j \right\rangle - \lambda G(\lambda) \\ &\quad + |\lambda|^2 \left\langle (A_B - \lambda)^{-1}F_i, v_j \right\rangle. \end{aligned}$$

Observe that $[\lambda G(\lambda)] = k[G](k) + \lim_{\varepsilon \searrow 0} i\varepsilon(G(k + i\varepsilon) + G(k - i\varepsilon))$. We shall prove later that

$$(49) \quad \lim_{\varepsilon \searrow 0} \varepsilon \left\langle (A_B - \lambda)^{-1}F_i, v_j \right\rangle = 0 \text{ for a.e. } k.$$

This implies $[\lambda G(\lambda)] = k[G](k)$ and that

$$[|\lambda|^2 \left\langle (A_B - \lambda)^{-1}F_i, v_j \right\rangle] = k^2 \left\langle [(A_B - \lambda)^{-1}]F_i, v_j \right\rangle;$$

hence, from (48), the formula

$$- \left[\left\langle (A_B - \lambda)^{-1}(\tilde{A}^* - \lambda)F_i, (A^* - \bar{\lambda})v_j \right\rangle \right] = \left\langle [(A_B - \lambda)^{-1}](\tilde{A}^* - k)F_i, (A^* - k)v_j \right\rangle$$

gives the jump of the left hand side of (46). We therefore have

$$(50) \quad -(\mu_i - k)(\bar{\mu}_j - k) \left\langle [(A_B - \lambda)^{-1}]F_i, v_j \right\rangle = \langle [M_B](k)f_i, w_j \rangle_{\mathcal{K}},$$

from which our result follows.

It remains to establish (49). Returning to (46) we have

$$-(\mu_i - \lambda)(\bar{\mu}_j - \lambda) \left\langle (A_B - \lambda)^{-1}F_i, v_j \right\rangle = (\bar{\mu}_j - \lambda) \left\langle F_i, v_j \right\rangle - \left\langle f_i, \tilde{\Gamma}_2 v_j \right\rangle_{\mathcal{H}} + \left\langle M_B(\lambda)f_i, w_j \right\rangle_{\mathcal{K}}$$

whence

$$\left| \left\langle (A_B - \lambda)^{-1}F_i, v_j \right\rangle \right| \leq \frac{|\bar{\mu}_j - \lambda| \left| \left\langle F_i, v_j \right\rangle \right| + \left| \left\langle f_i, \tilde{\Gamma}_2 v_j \right\rangle_{\mathcal{H}} \right| + \left| \langle M_B(\lambda)f_i, w_j \rangle_{\mathcal{K}} \right|}{|\mu_i - \lambda| |\bar{\mu}_j - \lambda|}.$$

Thus (49) is an immediate consequence of the hypothesis that $\varepsilon|\langle M_B(k \pm i\varepsilon)f, w \rangle| \rightarrow 0$ as $\varepsilon \searrow 0$, for a.e. $k \in \mathbb{R}$. \square

Theorem 6.5. *Let $\{f_i\}_{i \in I}$ and $\{w_j\}_{j \in \tilde{I}}$ be linearly independent vectors whose spans are dense in \mathcal{H} and \mathcal{K} respectively. Let $\{\mu_\ell\}_{\ell \in J}$ and $\{\tilde{\mu}_\nu\}_{\nu \in \tilde{J}}$ be collections of distinct strictly complex numbers dense in $\mathbb{C} \setminus \sigma(A_B)$ and $\mathbb{C} \setminus \sigma(A_B^*)$ respectively. Define $F_{i,\ell} = S_{\mu_\ell, B} f_i$, $v_{j,\nu} = \tilde{S}_{\tilde{\mu}_\nu, B^*} w_j$.*

- (1) *The collections $\{F_{i,\ell}\}_{i \in I, \ell \in J}$ and $\{v_{j,\nu}\}_{j \in \tilde{I}, \nu \in \tilde{J}}$ are both linearly independent and their spans are dense in $\overline{\mathcal{S}}$ and $\overline{\tilde{\mathcal{S}}}$ respectively.*
- (2) *For $N, M \in \mathbb{N}$ let $P_{N, \overline{\mathcal{S}}}$ and $P_{M, \overline{\tilde{\mathcal{S}}}}$ denote projections onto N - and M -dimensional subspaces of $\overline{\mathcal{S}}$ and $\overline{\tilde{\mathcal{S}}}$ respectively, spanned by N of the $F_{i,\ell}$ and M of the $v_{j,\nu}$ respectively, chosen such that $\lim_{N \rightarrow \infty} P_{N, \overline{\mathcal{S}}} = I$ and $\lim_{M \rightarrow \infty} P_{M, \overline{\tilde{\mathcal{S}}}} = I$, in the sense of strong convergence. Put*

$$E_1 = \left\{ k \in \mathbb{R} \mid [M_B](k) \text{ exists in the weak topology} \right\},$$

$$E_2 = \left\{ k \in \mathbb{R} \mid \lim_{\varepsilon \searrow 0} \varepsilon |\langle M_B(k \pm i\varepsilon)f_i, w_j \rangle| = 0 \text{ for all } i, j \right\}.$$

For any $k \in E_1 \cap E_2$ we have that $[P_{M, \overline{\tilde{\mathcal{S}}}}(A_B - \lambda)^{-1} P_{N, \overline{\mathcal{S}}}] (k)$ exists; moreover

$$\lim_{N, M \rightarrow \infty} \text{rank}([P_{M, \overline{\tilde{\mathcal{S}}}}(A_B - \lambda)^{-1} P_{N, \overline{\mathcal{S}}}]|_{\lambda=k})$$

exists and is equal to $\text{rank}([M_B](k))$.

Proof.

- (1) The fact that the closed linear spans of the sets $\{F_{i,\ell}\}_{i \in I, \ell \in J}$ and $\{v_{j,\nu}\}_{j \in \tilde{I}, \nu \in \tilde{J}}$ are $\overline{\mathcal{S}}$ and $\overline{\tilde{\mathcal{S}}}$ follows immediately from the definitions of $\overline{\mathcal{S}}$ and $\overline{\tilde{\mathcal{S}}}$ together with the fact that the closed linear spans of $\{f_i\}_{i \in I}$ and $\{w_j\}_{j \in \tilde{I}}$ are \mathcal{H} and \mathcal{K} respectively. It remains only to establish linear independence. Assume that there exist constants $\alpha_{i,\ell}$ such that $\sum_{i,\ell} \alpha_{i,\ell} F_{i,\ell} = 0$. This means that $\sum_{i,\ell} \alpha_{i,\ell} S_{\mu_\ell, B} f_i = 0$. Applying \tilde{A}^* k times yields

$$\sum_{i,\ell} \alpha_{i,\ell} \mu_\ell^k F_{i,\ell} = \sum_{\ell} \mu_\ell^k \sum_i \alpha_{i,\ell} F_{i,\ell} = 0, \quad k = 0, 1, 2, \dots$$

Since the μ_ℓ are distinct this yields $\sum_i \alpha_{i,\ell} F_{i,\ell} = 0$ for all ℓ . This means $S_{\mu_\ell, B} \sum_i \alpha_{i,\ell} f_i = 0$, and since $S_{\mu_\ell, B}$ has a left inverse this implies $\sum_i \alpha_{i,\ell} f_i = 0$. But the $\{f_i\}$ are linearly independent, so we deduce that $\alpha_{i,\ell} = 0$ for all i and ℓ .

- (2) Let $P_{N, \overline{\mathcal{S}}}$ denote projection onto N of the $F_{i,\ell}$ and $P_{M, \overline{\tilde{\mathcal{S}}}}$ denote projection onto M of the $v_{j,\nu}$, chosen in each case to be such that $P_{N, \overline{\mathcal{S}}}$ and $P_{M, \overline{\tilde{\mathcal{S}}}}$ converge strongly to the identity. Let $P_{N'}$ and $\tilde{P}_{M'}$ denote projections onto the spaces spanned by the corresponding f_i , of which there will be $N' \leq N$, and w_j , of which there will be $M' \leq M$. Since $k \in E_1 \cap E_2$ we may invoke (50) from the proof of Theorem 6.2 and deduce that

$$-(\mu_\ell - k)(\tilde{\mu}_\nu - k) \left\langle [P_{M, \overline{\tilde{\mathcal{S}}}}(A_B - \lambda)^{-1} P_{N, \overline{\mathcal{S}}}]_{\lambda=k} F_{i,\ell}, v_{j,\nu} \right\rangle = \left\langle [\tilde{P}_{M'} M_B(\lambda) P_{N'}]_{\lambda=k} f_i, w_j \right\rangle.$$

As $k \in \mathbb{R}$ we know that $\mu_\ell \neq k$ and $\tilde{\mu}_\nu \neq k$, so we define

$$X_{i,\ell} = (\mu_\ell - k) F_{i,\ell}, \quad Y_{j,\nu} = -(\tilde{\mu}_\nu - k) v_{j,\nu}.$$

The vectors $\{X_{i,\ell}\}$ and $\{Y_{j,\nu}\}$ are both linearly independent, and for each $\ell \in J, \nu \in \tilde{J}$ we have

$$\left\langle [P_{M, \overline{\tilde{\mathcal{S}}}}(A_B - \lambda)^{-1} P_{N, \overline{\mathcal{S}}}]_{\lambda=k} X_{i,\ell}, Y_{j,\nu} \right\rangle = \left\langle [\tilde{P}_{M'} M_B(\lambda) P_{N'}]_{\lambda=k} f_i, w_j \right\rangle.$$

Define M_1 to be the matrix with entries

$$\left\langle [P_{M,\bar{S}}(A_B - \lambda)^{-1}P_{N,\bar{S}}]_{\lambda=k} X_{i,\ell}, Y_{j,\nu} \right\rangle,$$

ordered by incrementing ℓ and ν before i and j , and M_2 to be the matrix with entries $\left\langle [\tilde{P}_{M'}M_B(\lambda)P_{N'}]_{\lambda=k} f_i, w_j \right\rangle$. It follows from the definition of the Kronecker product that

$$M_1 = M_2 \otimes E$$

in which E is a matrix whose entries are all equal to 1. By consideration of the singular values of the Kronecker product (E has only one non-zero singular value) it follows that M_1 and M_2 have the same rank, and hence that

$$\text{rank}([P_{M,\bar{S}}(A_B - \lambda)^{-1}P_{N,\bar{S}}]_{\lambda=k}) = \text{rank}([\tilde{P}_{M'}M_B(\lambda)P_{N'}]_{\lambda=k}).$$

If we define

$$\text{rank}([P_{\bar{S}}(A_B - \lambda)^{-1}P_{\bar{S}}]_{\lambda=k}) := \lim_{M,N \rightarrow \infty} \text{rank}([P_{M,\bar{S}}(A_B - \lambda)^{-1}P_{N,\bar{S}}]_{\lambda=k})$$

and exploit the fact that the $\{f_i\}_{i \in I}$ and $\{v_j\}_{j \in \bar{I}}$ exhaust \mathcal{H} and \mathcal{K} respectively, then it follows that

$$\text{rank}([P_{\bar{S}}(A_B - \lambda)^{-1}P_{\bar{S}}]_{\lambda=k}) = \text{rank}([M_B(\lambda)]_{\lambda=k}).$$

□

As a final remark, we mention that $\text{rank}([P_{\bar{S}}(A_B - \lambda)^{-1}P_{\bar{S}}]_{\lambda=k})$ is the multiplicity of the absolutely continuous spectrum of $A_B|_{\bar{S}}$.

7. FRIEDRICHS MODEL: RECONSTRUCTION OF $M_B(\lambda)$ FROM ONE RESTRICTED RESOLVENT $(A_B - \lambda)^{-1}|_{\bar{S}}$

We have now finished our abstract considerations and the remainder of the paper is devoted to a detailed analysis of the Friedrichs model.

In this section we show how to reconstruct $M_B(\lambda)$ explicitly from the restricted resolvent. The fact that even the bordered resolvent determines $M_B(\lambda)$ uniquely was proved in the abstract setting in Theorem 4.1, but of course methods of reconstruction depend on the concrete operators under scrutiny.

We introduce the notation $\hat{\cdot}$ for the Cauchy or Borel transform given by

$$(51) \quad \hat{\phi}(\lambda) = \left\langle \frac{1}{t - \lambda}, \phi \right\rangle, \quad \hat{\psi}(\lambda) = \left\langle \frac{1}{t - \lambda}, \bar{\psi} \right\rangle$$

and $P_{\pm} : L^2(\mathbb{R}) \rightarrow H_2^{\pm}(\mathbb{R})$ for the Riesz projections given by

$$(52) \quad P_{\pm} f(k) = \pm \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0} \hat{f}(k \pm i\varepsilon) = \pm \frac{1}{2\pi i} \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \frac{f(x)}{x - (k \pm i\varepsilon)} dx,$$

where the limit is to be understood in $L^2(\mathbb{R})$ (see [28]). Here, $H_p^+(\mathbb{R})$ and $H_p^-(\mathbb{R})$ denote the Hardy spaces of boundary values of p -integrable functions in the upper and lower complex half-plane, respectively. To simplify notation, we also sometimes write $(\hat{f})_{\pm}(k) = \hat{f}(k \pm i0) := 2\pi i P_{\pm} f(k)$.

Theorem 7.1. *For the Friedrichs model, assume that $(A_B - \lambda)^{-1}|_{\bar{S}}$ is known for all $\lambda \in \rho(A_B) \setminus \mathbb{R}$. Then $M_B(\lambda)$ can be recovered.*

Remark 7.2. We assume that $(A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}}$ is known for all $\lambda \in \rho(A_B) \setminus \mathbb{R}$, though it is certainly sufficient to know it at one point in each connected component of $\mathbb{C} \setminus \sigma(A_B)$. If $\sigma(A_B)$ does not cover all of either half-plane \mathbb{C}_{\pm} then it is enough to know $(A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}}$ at two points, one in each of \mathbb{C}_{\pm} . If, additionally, $\sigma(A_B)$ does not cover \mathbb{R} , then it suffices to know $(A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}}$ for just one value of λ .

Proof of Theorem 7.1. 1. Recovering the function ψ . Take non-zero $g \in \overline{\mathcal{S}}$ and $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup \sigma(A_B))$. Observe that (35) may be rewritten in the form

$$(53) \quad f(x) - \frac{g(x)}{x - \lambda} - \frac{c_f}{x - \lambda} = \frac{\psi(x)}{x - \lambda} A(\lambda),$$

in which

$$A(\lambda) = -\frac{1}{D(\lambda)} \left[\left\langle \frac{g}{t - \lambda}, \phi \right\rangle + c_f \left\langle \frac{1}{t - \lambda}, \phi \right\rangle \right]$$

and $D(\lambda)$ is given by (33). The left hand side of (53) is known as a function of λ , at least for $g \in \overline{\mathcal{S}}$. To determine ψ **up to a scalar multiple** it is therefore sufficient to find g and λ so that $A(\lambda)$ is non-zero: in other words, find g such that the function $A(\cdot)$ is not identically zero.

We proceed by contradiction. Suppose we have a non-trivial Friedrichs model (i.e. neither ϕ nor ψ is identically zero). If $A(\cdot)$ is identically zero then multiplying by $M_B(\lambda)^{-1}$ from (34) and using (36) we obtain

$$(54) \quad \begin{aligned} & \left[i\pi \text{sign}(\Im \lambda) - \frac{1}{D(\lambda)} \left\langle \frac{1}{t - \lambda}, \phi \right\rangle \left\langle \frac{1}{t - \lambda}, \bar{\psi} \right\rangle - B \right] \left\langle \frac{g}{t - \lambda}, \phi \right\rangle \\ & + \left[- \left\langle \frac{1}{t - \lambda}, \bar{g} \right\rangle + \frac{1}{D(\lambda)} \left\langle \frac{g}{t - \lambda}, \phi \right\rangle \left\langle \frac{1}{t - \lambda}, \bar{\psi} \right\rangle \right] \left\langle \frac{1}{t - \lambda}, \phi \right\rangle \equiv 0, \end{aligned}$$

from which it follows

$$(55) \quad (i\pi \text{sign}(\Im \lambda) - B) \left\langle \frac{g}{t - \lambda}, \phi \right\rangle - \left\langle \frac{1}{t - \lambda}, \bar{g} \right\rangle \left\langle \frac{1}{t - \lambda}, \phi \right\rangle \equiv 0.$$

For all non-real μ such that $D(\mu)$ is nonzero (this is true for a.e. non-real μ by analyticity), there exists $g \in \overline{\mathcal{S}}$ in the range of the solution operator $S_{\mu, B}$. We know from (32) that such g have the form

$$(56) \quad g(x) = \frac{1}{x - \mu} - \frac{1}{D(\mu)} \left\langle \frac{1}{t - \mu}, \phi \right\rangle \frac{\psi(x)}{x - \mu},$$

though we do not know the function ψ or the value of $\frac{1}{D(\mu)} \left\langle \frac{1}{t - \mu}, \phi \right\rangle$. Substituting (56) into (55) yields

$$(57) \quad \begin{aligned} & (i\pi \text{sign}(\Im \lambda) - B) \left[\left\langle \frac{1}{(t - \mu)(t - \lambda)}, \phi \right\rangle - \frac{1}{D(\mu)} \left\langle \frac{1}{t - \mu}, \phi \right\rangle \left\langle \frac{\psi}{(t - \mu)(t - \lambda)}, \phi \right\rangle \right] \\ & \equiv \left\langle \frac{1}{t - \lambda}, \phi \right\rangle \left[\left\langle \frac{1}{(t - \lambda)(t - \mu)}, \mathbf{1} \right\rangle - \frac{1}{D(\mu)} \left\langle \frac{1}{t - \mu}, \phi \right\rangle \left\langle \frac{1}{(t - \lambda)(t - \mu)}, \bar{\psi} \right\rangle \right]. \end{aligned}$$

If we use the identity

$$(58) \quad \frac{\lambda - \mu}{(t - \lambda)(t - \mu)} = \frac{1}{t - \lambda} - \frac{1}{t - \mu}$$

and use the notations from (51) then multiplying by $(\lambda - \mu)$, (57) becomes

$$(59) \quad (i\pi \text{sign}(\Im \lambda) - B) \left[\widehat{\phi}(\lambda) - \widehat{\phi}(\mu) - \frac{1}{D(\mu)} \widehat{\phi}(\mu)(D(\lambda) - D(\mu)) \right] \\ \equiv \widehat{\phi}(\lambda) \left[\int_{\mathbb{R}} \frac{\lambda - \mu}{(t - \lambda)(t - \mu)} dt - \frac{\widehat{\phi}(\mu)}{D(\mu)} (\widehat{\psi}(\lambda) - \widehat{\psi}(\mu)) \right].$$

Performing the integral for the case in which $\Im \lambda \cdot \Im \mu < 0$, we obtain

$$(60) \quad (i\pi \text{sign}(\Im \lambda) - B) \left[\widehat{\phi}(\lambda) - \frac{D(\lambda)}{D(\mu)} \widehat{\phi}(\mu) \right] \equiv \widehat{\phi}(\lambda) \left[\pm 2\pi i - \frac{\widehat{\phi}(\mu)}{D(\mu)} (\widehat{\psi}(\lambda) - \widehat{\psi}(\mu)) \right].$$

Fix λ and let $\mu \rightarrow i\infty$, so that $D(\mu) \rightarrow 1$ and $\widehat{\phi}(\mu) \rightarrow 0$. This yields

$$(61) \quad (i\pi \text{sign}(\Im \lambda) - B) \widehat{\phi}(\lambda) \equiv \pm 2\pi i \widehat{\phi}(\lambda).$$

If, on the other hand, we consider $\Im \lambda \cdot \Im \mu > 0$ in (59) then the value of the integral is zero, and we obtain, upon letting $\mu \rightarrow i\infty$,

$$(62) \quad (i\pi \text{sign}(\Im \lambda) - B) \widehat{\phi}(\lambda) \equiv 0.$$

Equations (61,62) together imply that $\widehat{\phi}$ is identically zero, and hence so is ϕ . In this case the function ψ is irrelevant and so our Friedrichs model is trivial, a contradiction. Thus (53) determines ψ up to a constant multiple. We may choose this (non-zero) multiple arbitrarily, since ϕ can be rescaled if necessary to obtain the correct Friedrichs model.

2. Recovering the boundary condition parameter B . Returning to the parameter c_f in (36) and using the notation (51), we have

$$\left[i\pi \text{sign}(\Im \lambda) - B - \frac{1}{D(\lambda)} \widehat{\phi}(\lambda) \widehat{\psi}(\lambda) \right] c_f = \left[- \left\langle \frac{1}{t - \lambda}, \bar{g} \right\rangle + \frac{1}{D(\lambda)} \left\langle \frac{g}{t - \lambda}, \phi \right\rangle \left\langle \frac{1}{t - \lambda}, \bar{\psi} \right\rangle \right] \\ = \left[- \left\langle \frac{1}{t - \lambda}, \bar{g} \right\rangle + O(\|g\|_2 |\Im \lambda|^{-3/2}) \right],$$

as $\Im \lambda \rightarrow \infty$, and uniformly in g . Now choose an element

$$(63) \quad g(x) \equiv g_\mu(x) := \frac{1}{x - \mu} - \sigma(\mu) \frac{\psi(x)}{x - \mu},$$

$\mu \in \mathbb{C} \setminus \mathbb{R}$, $D(\mu) \neq 0$, with some $\sigma(\mu) = O(|\Im \mu|^{-1/2})$. We know that such $\sigma(\mu)$ exists, and indeed may be chosen as $\widehat{\phi}(\mu)/D(\mu)$, but we do not yet know ϕ and therefore do not claim that our particular choice of σ is given by this formula. We fix some choice of σ , so that $g = g_\mu$ is determined and c_f is known as a function of λ and μ . We have

$$(i\pi \text{sign}(\Im \lambda) - B + O(|\Im \lambda|^{-1})) c_f \\ = \left[- \left\langle \frac{1}{t - \lambda}, \frac{1}{t - \bar{\mu}} \right\rangle + \sigma(\mu) \left\langle \frac{1}{t - \lambda}, \frac{\bar{\psi}}{t - \bar{\mu}} \right\rangle + O(|\Im \lambda|^{-3/2}) \|g_\mu\|_2 \right] \\ = - \int_{\mathbb{R}} \frac{1}{(t - \lambda)(t - \mu)} dt + O(|\Im \mu|^{-3/2}) O(|\Im \lambda|^{-1/2}) \\ + O(|\Im \lambda|^{-3/2}) \left(O(|\Im \mu|^{-1/2}) + \|\psi\|_2 \frac{|\sigma(\mu)|}{|\Im \mu|} \right).$$

Assuming that $\Im \lambda \cdot \Im \mu < 0$, we know that

$$-\int_{\mathbb{R}} \frac{1}{(t-\lambda)(t-\mu)} dt = \frac{\pm 2\pi i}{\lambda - \mu}.$$

Put $\lambda = -\mu$ and letting $\Im \mu \rightarrow \infty$, we obtain

$$(i\pi \text{sign}(\Im \lambda) - B)c_f = \frac{\pm 2\pi i}{2\lambda} + O(|\lambda|^{-2}).$$

For one choice of $\text{sign}(\Im \lambda)$ at least, $i\pi \text{sign}(\Im \lambda) - B \neq 0$ and so we can recover B from the asymptotic behaviour of c_f as $\Im \lambda \rightarrow \infty$.

3. Recovering $\widehat{\phi}(\lambda)/D(\lambda)$. Once again we choose $g = g_\mu$ of the form (63). Returning to (53) and indicating the μ -dependence of f by writing $f = f_\mu = (A_B - \lambda)^{-1}g_\mu$, we have

$$(A_B - \lambda)^{-1}g_\mu - \frac{g_\mu(x)}{x - \lambda} - \frac{c_{f_\mu}(\lambda)}{x - \lambda} = -\frac{\psi(x)}{x - \lambda} \frac{1}{D(\lambda)} \left[\left\langle \frac{g_\mu}{t - \lambda}, \phi \right\rangle + c_{f_\mu}(\lambda) \left\langle \frac{1}{t - \lambda}, \phi \right\rangle \right].$$

Since the left hand side of this equation is known and since ψ is known, this implies that

$$\frac{1}{D(\lambda)} \left[\left\langle \frac{g_\mu}{t - \lambda}, \phi \right\rangle + c_{f_\mu}(\lambda) \left\langle \frac{1}{t - \lambda}, \phi \right\rangle \right]$$

is known. Substituting the known choice of g_μ we discover that

$$\frac{1}{D(\lambda)} \left[\left\langle \frac{1}{(t - \lambda)(t - \mu)}, \phi \right\rangle - \sigma(\mu) \left\langle \frac{\psi}{(t - \lambda)(t - \mu)}, \phi \right\rangle + c_{f_\mu}(\lambda) \left\langle \frac{1}{t - \lambda}, \phi \right\rangle \right] (\lambda - \mu)$$

is known too. Using identity (58) this means that

$$(64) \quad \frac{1}{D(\lambda)} \left[\widehat{\phi}(\lambda) - \widehat{\phi}(\mu) - \sigma(\mu)(D(\lambda) - D(\mu)) + (\lambda - \mu)c_{f_\mu}(\lambda)\widehat{\phi}(\lambda) \right]$$

is known. We shall now fix λ and let $\Im \mu \rightarrow \infty$, for which purpose we need to know how $(\lambda - \mu)c_{f_\mu}(\lambda)$ will behave. From (36), we have

$$(65) \quad c_{f_\mu}(\lambda)(\lambda - \mu) = (\lambda - \mu)M_B(\lambda) \left[-\left\langle \frac{1}{t - \lambda}, \frac{1}{t - \bar{\mu}} \right\rangle + \sigma(\mu) \left\langle \frac{1}{t - \lambda}, \frac{\bar{\psi}}{t - \bar{\mu}} \right\rangle + \frac{\widehat{\psi}(\lambda)}{D(\lambda)} \left\{ \left\langle \frac{1}{(t - \lambda)(t - \mu)}, \phi \right\rangle - \sigma(\mu) \left\langle \frac{\psi}{(t - \lambda)(t - \mu)}, \phi \right\rangle \right\} \right].$$

Choosing $\mu \neq \lambda$ with $\Im \lambda \cdot \Im \mu > 0$ causes the integral term $\left\langle \frac{1}{t - \lambda}, \frac{1}{t - \bar{\mu}} \right\rangle$ to vanish. This yields

$$(66) \quad c_{f_\mu}(\lambda) = M_B(\lambda) \left[\sigma(\mu)(\widehat{\psi}(\lambda) - \widehat{\psi}(\mu)) + \frac{\widehat{\psi}(\lambda)}{D(\lambda)}(\widehat{\phi}(\lambda) - \widehat{\phi}(\mu)) - \sigma(\mu)(D(\lambda) - D(\mu)) \right] \rightarrow M_B(\lambda) \frac{\widehat{\psi}(\lambda)}{D(\lambda)} \widehat{\phi}(\lambda), \quad \Im \mu \rightarrow \infty.$$

Letting $\Im \mu \rightarrow \infty$ in (64) therefore yields that

$$(67) \quad \frac{1}{D(\lambda)} \left[\widehat{\phi}(\lambda) + M_B(\lambda) \frac{\widehat{\psi}(\lambda)}{D(\lambda)} \widehat{\phi}(\lambda)^2 \right]$$

is known. However from (34) we have

$$M_B(\lambda) = \left[i\pi \text{sign}(\Im \lambda) - \frac{1}{D(\lambda)} \widehat{\phi}(\lambda) \widehat{\psi}(\lambda) - B \right]^{-1}$$

and so the known quantity appearing in (67) is

$$M_B(\lambda) \frac{\widehat{\phi}(\lambda)}{D(\lambda)} [i\pi \operatorname{sign}(\Im \lambda) - B].$$

This means that

$$\alpha := M_B(\lambda) \frac{\widehat{\phi}(\lambda)}{D(\lambda)}$$

is known, and simple algebra shows that

$$(68) \quad \frac{\widehat{\phi}(\lambda)}{D(\lambda)} (1 + \alpha \widehat{\psi}(\lambda)) = \alpha (i\pi \operatorname{sign}(\Im \lambda) - B),$$

which determines $\frac{\widehat{\phi}(\lambda)}{D(\lambda)}$ and hence $M_B(\lambda)$ provided the factor $1 + \alpha \widehat{\psi}(\lambda)$ is not identically zero; equivalently, provided $i\pi \operatorname{sign}(\Im \lambda) - B$ is not zero.

We are therefore left to rule out just one pathological case: the case in which $B = i\pi \operatorname{sign}(\Im \lambda)$ in one half-plane and $\widehat{\phi}\widehat{\psi} \equiv 0$ in the same half-plane. This can only happen if $M_B(\lambda)^{-1}$ is zero in this half-plane, which means that every point in the half-plane is an eigenvalue of A_B and the corresponding g_λ given by

$$g_\lambda(x) = \frac{1}{x - \lambda} - \frac{\widehat{\phi}(\lambda)}{D(\lambda)} \frac{\psi(x)}{x - \lambda} = \frac{1}{x - \lambda}$$

belongs to $L^2(\mathbb{R})$ and also satisfies the conditions to lie in the domain of A_B :

$$i\pi \operatorname{sign}(\Im \lambda) = i\pi \operatorname{sign}(\Im \lambda) - \frac{\widehat{\phi}(\lambda)\widehat{\psi}(\lambda)}{D(\lambda)} = \Gamma_1 g_\lambda = B \Gamma_2 g_\lambda = B$$

(see (6.16) in [13]). □

Remark 7.3. (Uniqueness of g_μ). *An alternative approach can be found by examining the uniqueness of the function g_μ in $\overline{\mathcal{S}}$ defined in (63). If we know that the choice of $\sigma(\mu)$ is unique then we can immediately determine $\widehat{\phi}(\mu)/D(\mu)$, which must be equal to $\sigma(\mu)$. This is determined by g_μ if g_μ is unique with its required properties. We examine this now.*

Definition 7.4. *The non-uniqueness set is the set*

$$(69) \quad \Omega = \left\{ \mu \in \mathbb{C} \setminus \mathbb{R} \mid \exists \sigma_1(\mu) \neq \sigma_2(\mu) : \frac{1}{x - \mu} + \sigma_j(\mu) \frac{\psi(x)}{x - \mu} \in \overline{\mathcal{S}}, \quad j = 1, 2 \right\}.$$

Equivalently,

$$\Omega = \left\{ \mu \in \mathbb{C} \setminus \mathbb{R} \mid \frac{1}{x - \mu} \in \overline{\mathcal{S}} \text{ and } \frac{\psi(x)}{x - \mu} \in \overline{\mathcal{S}} \right\}.$$

We also let $\Omega_\pm = \mathbb{C}_\pm \cap \Omega$. We can ignore the condition $D(\mu) \neq 0$ since it can be removed by taking a closure. We can also assume that $\overline{\mathcal{S}} \neq L^2(\mathbb{R})$ since otherwise we know the whole resolvent $(A_B - \lambda)^{-1}$, which means we know A_B and hence M_B . We consider two cases in \mathbb{C}_+ (the situation in \mathbb{C}_- is similar):

- (I): $\mathbb{C}_+ \setminus \Omega_+$ has measure 0;
- (II): $\mathbb{C}_+ \setminus \Omega_+$ has positive measure.

In case (II) the uniqueness set in \mathbb{C}_+ , where we can recover $\widehat{\phi}(\mu)/D(\mu)$ immediately from g_μ , will have an accumulation point in \mathbb{C}_+ and thus $\widehat{\phi}(\mu)/D(\mu)$ is uniquely determined in \mathbb{C}_+ , by analyticity.

In case (I) we have that for almost all $\mu \in \mathbb{C}_+$, the function $x \mapsto (x - \mu)^{-1}$ lies in $\overline{\mathcal{S}}$. However $\bigvee_{\Im \mu > 0} \frac{1}{x - \mu}$ is the Hardy space H_2^- , and hence $\overline{\mathcal{S}} \supseteq H_2^-$. Consider the situation in \mathbb{C}_- . If we are in the case $|\Omega_-| > 0$ then

$$\overline{\mathcal{S}} \supset \bigvee_{\mu \in \Omega_-} \frac{1}{x - \mu} = H_2^+,$$

and so we have proved the following.

Lemma 7.5. (1) *If $\mathbb{C}_\pm \setminus \Omega_\pm$ has measure zero, then $\overline{\mathcal{S}}$ contains H_\mp^2 , respectively.*

(2) *If $\mathbb{C}_\pm \setminus \Omega_\pm$ has positive measure then one can recover $\widehat{\phi}(\mu)/D(\mu)$ uniquely, for $\mu \in \mathbb{C}_\pm$.*

Corollary 7.6. *Assume that the function $\widehat{\phi}(\mu)/D(\mu)$ in \mathbb{C}_+ coincides with the analytic continuation of $\widehat{\phi}(\mu)/D(\mu)$ in \mathbb{C}_- . (This happens, for instance, if ϕ has compact support or is zero on an interval.) Then either $\overline{\mathcal{S}} = L^2(\mathbb{R})$ or we can reconstruct $\widehat{\phi}(\mu)/D(\mu)$ in $\mathbb{C} \setminus \mathbb{R}$ uniquely from $(A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}}$.*

8. DETERMINING $\overline{\mathcal{S}}$ FOR THE FRIEDRICHS MODEL

This section is devoted to a detailed analysis of the space $\overline{\mathcal{S}}$ for the Friedrichs model. We shall demonstrate how different aspects of complex analysis are brought into the problem of determining $\overline{\mathcal{S}}$ and we compute the defect number

$$\text{def}(\overline{\mathcal{S}}) = \dim(\mathcal{S}^\perp)$$

for various different choices of the functions ϕ and ψ which determine the model. The proofs are almost all in the appendix, as the calculations are sometimes elaborate.

We first give a characterisation of the space $\overline{\mathcal{S}}$, or more precisely, its orthogonal complement.

Proposition 8.1. *Let P_\pm be the Riesz projections defined in (52) and $D(\lambda)$ be as in (33). Denote by $D_\pm(\lambda)$ its restriction to \mathbb{C}_\pm and by D_\pm the boundary values of these functions on \mathbb{R} (which exist a.e., cf. [28, 41]).*

(1) *Let $\phi, \psi \in L^2$.*

$$(70) \quad g \in \overline{\mathcal{S}}^\perp \iff \begin{cases} P_+ \bar{g} - \frac{2\pi i}{D_+}(P_+ \bar{\phi}) P_+(\psi \bar{g}) = 0, \\ P_- \bar{g} + \frac{2\pi i}{D_-}(P_- \bar{\phi}) P_-(\psi \bar{g}) = 0. \end{cases}$$

$$(71) \quad \iff \begin{cases} \text{(i)} \quad \frac{(P_+ \bar{\phi}) P_+(\psi \bar{g})}{D_+} \in H_2^+, \\ \text{(ii)} \quad \frac{(P_- \bar{\phi}) P_-(\psi \bar{g})}{D_-} \in H_2^-, \\ \text{(iii)} \quad \bar{g} - \frac{2\pi i}{D_+}(P_+ \bar{\phi}) P_+(\psi \bar{g}) + \frac{2\pi i}{D_-}(P_- \bar{\phi}) P_-(\psi \bar{g}) = 0 \text{ (a.e.)}. \end{cases}$$

(2) *If $\phi \in L^2, \psi \in L^2 \cap L^\infty$ or $\phi, \psi \in L^2 \cap L^4$, then*

$$(72) \quad g \in \overline{\mathcal{S}}^\perp \iff [D_+ - 2\pi i(P_+ \bar{\phi})\psi] \bar{g} = 2\pi i \bar{\phi}[\psi P_- \bar{g} - P_-(\psi \bar{g})] \text{ (a.e.)},$$

$$(73) \quad \iff [D_+ - 2\pi i(P_+ \bar{\phi})\psi] \bar{g} = 2\pi i \bar{\phi}[-\psi P_+ \bar{g} + P_+(\psi \bar{g})] \text{ (a.e.)},$$

$$(74) \quad \iff [D_+ - 2\pi i(P_+ \bar{\phi})\psi] \bar{g} = 2\pi i \bar{\phi}[P_+(\psi P_- \bar{g}) - P_-(\psi P_+ \bar{g})] \text{ (a.e.)}.$$

- Remark 8.2.** (1) *The second part of the proposition allows us to replace all three conditions (i)-(iii) in (71) with a single pointwise condition under mild extra assumptions on ϕ and/or ψ .*
- (2) *Note that the operator $[P_+(\psi P_- \bar{g}) - P_-(\psi P_+ \bar{g})]$ in the last characterisation of $\bar{\mathcal{S}}^\perp$ is the difference of two Hankel operators.*

As an immediate consequence of (72), we get

Theorem 8.3. *Assume $\phi \in L^2, \psi \in L^2 \cap L^\infty$ or $\phi, \psi \in L^2 \cap L^4$. Define the operator L on $L^2(\mathbb{R})$ by*

$$(75) \quad Lu = [-P_+(\psi \bar{\phi}) + P_+(\bar{\phi})\psi]u + \bar{\phi}[\psi P_- - P_- \psi]u$$

with the maximal domain $D(L) = \{u \in L^2(\mathbb{R}) : Lu \in L^2(\mathbb{R})\}$.¹ Then $\bar{\mathcal{S}} \neq L^2(\mathbb{R})$ iff $1/(2\pi i) \in \sigma_p(L)$ and $\bar{\mathcal{S}}^\perp = \ker(L - 1/(2\pi i))$.

Furthermore, let $\eta \in L^\infty(\mathbb{R})$ be a function such that $\eta(k) \neq 0$ a.e. and $\eta[-P_+(\psi \bar{\phi}) + P_+(\bar{\phi})\psi], \eta\psi \bar{\phi}, \eta \bar{\phi} \in L^\infty(\mathbb{R})$. Define the operator \mathcal{L} on $L^2(\mathbb{R})$ by

$$(76) \quad \mathcal{L}u = \eta \left[-\frac{1}{2\pi i} - P_+(\psi \bar{\phi}) + P_+(\bar{\phi})\psi \right] u + \eta \bar{\phi}[\psi P_- - P_- \psi]u$$

with dense domain $D(\mathcal{L}) = \{u \in L^2(\mathbb{R}) : \eta \bar{\phi} P_-(\psi u) \in L^2(\mathbb{R})\}$. Then $\bar{\mathcal{S}} \neq L^2(\mathbb{R})$ iff $0 \in \sigma_p(\mathcal{L})$. Moreover, $\bar{\mathcal{S}}^\perp = \ker \mathcal{L}$. Note that if $\psi \in L^\infty$, then $D(\mathcal{L}) = L^2(\mathbb{R})$.

Remark 8.4. *Replacing ψ by $\alpha\psi$, we denote the corresponding detectable subspace by $\bar{\mathcal{S}}_\alpha$. Then, under the conditions in the second part of Proposition 8.1, we get $g \in \mathcal{S}_\alpha^\perp$ iff*

$$\frac{1}{2\pi i \alpha} \bar{g} = [-P_+(\psi \bar{\phi}) + P_+(\bar{\phi})\psi] \bar{g} + \bar{\phi}[P_+ \psi P_- - P_- \psi P_+] \bar{g} = L \bar{g},$$

where the right hand side is the sum of a multiplication operator and the difference of two Hankel operators multiplied by $\bar{\phi}$. As in the theorem, we then get $\mathcal{S}_\alpha^\perp \neq \{0\}$ iff $1/(2\pi i \alpha) \in \sigma_p(L)$ and \mathcal{S}_α^\perp is given by the corresponding kernel.

We now consider various special cases that illustrate the different situations that can arise. The proofs of the results can be found in the appendix.

8.1. Results depending on the support of ϕ and ψ .

8.1.1. *The case of disjoint supports.* In this part we assume that $\phi \cdot \psi = 0$ almost everywhere, in particular, $D(\lambda) \equiv 1$, and that either

$$(77) \quad \phi \in L^2 \text{ and } \psi \in L^2 \cap L^\infty \text{ or } \phi, \psi \in L^2 \cap L^4.$$

In some cases (which will be mentioned in the text), we will require the slightly stronger condition

$$(78) \quad \phi \in L^2 \cap L^{2+\varepsilon} \text{ for some } \varepsilon > 0 \text{ and } \psi \in L^2 \cap L^\infty \text{ or } \phi, \psi \in L^2 \cap L^4.$$

Theorem 8.5. *Define the sets $\Omega_\psi = \{x \in \mathbb{R} : \psi(x) \neq 0\}$, $\Omega_\phi = \{x \in \mathbb{R} : \phi(x) \neq 0\}$ ² and $\Omega = \mathbb{R} \setminus (\Omega_\psi \cup \Omega_\phi)$. Then $g \in \bar{\mathcal{S}}^\perp$ iff $g \in L^2$ and*

$$(79) \quad \chi_{\Omega_\phi} \bar{g}(k) = \widehat{\bar{\phi}(k) \psi \chi_{\Omega_\psi} \bar{g}(k + i0)},$$

¹Under our assumptions, for any $u \in L^2$ we have $Lu \in L^1$ (where we mean the expression L in (75) rather the operator L).

²Note that the sets Ω_ψ, Ω_ϕ are only defined up to a set of Lebesgue measure zero, but this is sufficient for our purpose. Also, they may be much smaller than the support of the functions.

$$(80) \quad \chi_{\Omega_\psi} \bar{g}(k) \left(1 - \widehat{\bar{\phi}}(k - i0) \psi(k) \right) = 0,$$

$$(81) \quad g|_\Omega = 0.$$

Let $\Omega_{\psi,0} = \{k \in \mathbb{R} : \widehat{\bar{\phi}}(k - i0) \psi(k) = 1\}$. Then, additionally,

(i) if $\Omega_{\psi,0}$ has zero measure, then $\bar{\mathcal{S}} = L^2(\mathbb{R})$.

(ii) Assume (78). If $\Omega_{\psi,0}$ has non-zero measure, then $\bar{\mathcal{S}}^\perp \neq \{0\}$. Moreover, we have that

$$\bar{\mathcal{S}} \subseteq \{f \in L^2(\mathbb{R}) : f = \psi \left(\widehat{f\bar{\phi}} \right)_- \text{ on } \Omega_{\psi,0}\}.$$

(iii) Assume $\phi, \psi \in L^2 \cap L^\infty$. Then

$$\bar{\mathcal{S}} = \{f \in L^2(\mathbb{R}) : f = \psi \left(\widehat{f\bar{\phi}} \right)_- \text{ on } \Omega_{\psi,0}\}.$$

Remark 8.6. (1) Note that, from (79) we have that $g|_{\Omega_\psi}$ completely determines $g|_{\Omega_\phi}$.

(2) Condition (79) gives an additional restriction on $g|_{\Omega_\psi}$, requiring $\bar{\phi}(k) \widehat{\psi \chi_{\Omega_\psi} \bar{g}}(k + i0) \in L^2(\Omega_\phi)$.

(3) The condition (78) implies (via the Hölder inequality and boundedness of $P_+ : L^p \rightarrow L^p$ for $1 < p < \infty$) that for $g \in (L^2 \cap L^\infty)(\Omega_\psi)$ we have $\bar{\phi}(k) \widehat{\psi \chi_{\Omega_\psi} \bar{g}}(k + i0) \in L^2(\Omega_\phi)$.

Example 8.7. Let I and I' be disjoint closed intervals in \mathbb{R} . Choose $\phi \in L^\infty$ with $\text{supp } \phi \in I$ such that

$$\int_{\mathbb{R}} \frac{\bar{\phi}(x)}{x - k} dx \neq 0 \quad \text{for } k \in I'.$$

Define ψ by

$$\psi(k) = \begin{cases} \left(\int_{\mathbb{R}} \frac{\bar{\phi}(x)}{x - k} dx \right)^{-1}, & k \in I' \\ 0 & \text{otherwise.} \end{cases}$$

Then $\psi \in L^\infty$, $\phi \cdot \psi = 0$ and $\widehat{\bar{\phi}}(k - i0) \psi(k) = 1$. Therefore, by Theorem 8.5, (ii), $\bar{\mathcal{S}}^\perp \neq \{0\}$.

Theorem 8.8. Assume (78). Then $\bar{\mathcal{S}} = L^2 \iff 1 - \psi(\widehat{\bar{\phi}})(k + i0) \neq 0$ for a.e. $k \in \mathbb{R}$. Moreover,

$$\text{def } \mathcal{S} = \dim \mathcal{S}^\perp = \begin{cases} 0 & \text{if } \bar{\mathcal{S}} = L^2, \\ \infty & \text{otherwise.} \end{cases}$$

We have the following results on complete detectability, ie. $\bar{\mathcal{S}} = L^2(\mathbb{R})$.

Theorem 8.9. (1) Complete detectability is generic in the following sense. Replace ψ by $\alpha\psi$ for $\alpha \in \mathbb{C}$ (or ϕ by $\bar{\alpha}\phi$), then for all α outside a countable set E_0 we have $\bar{\mathcal{S}}_\alpha = L^2(\mathbb{R})$, where $\bar{\mathcal{S}}_\alpha$ is as defined in Remark 8.4.

(2) For small perturbations, we have that if $\psi \in L^\infty$ and $P_+\phi$ or $P_-\phi \in L^\infty$, replacing ψ by $\alpha\psi$ where $\alpha \in \mathbb{C}$, we get $\bar{\mathcal{S}}_\alpha = L^2(\mathbb{R})$ for sufficiently small $|\alpha|$.

We next analyse a specific example with disjoint supports where we consider some questions in more detail.

Theorem 8.10. Assume I and I' are disjoint closed intervals such that I' lies to the left of I . Let

$$\phi = \chi_I \quad \text{and} \quad \psi(x) = \chi_{I'}(x) \cdot \left(\int_I \frac{dt}{t - x} \right)^{-1}.$$

(1) In this case $\bar{\mathcal{S}} \neq L^2(\mathbb{R})$ with $\text{def } \bar{\mathcal{S}} = \infty$, while $\bar{\bar{\mathcal{S}}} = L^2(\mathbb{R})$.

(2) The jump of the M -function across the real axis at k is given by

$$[M_B^{-1}(k)] = \begin{cases} 2\pi i, & k \in \mathbb{R} \setminus (I \cup I'), \\ 2\pi i(1 - \widehat{\psi}(k)), & k \in I, \\ 0, & k \in I'. \end{cases}$$

Moreover, the bordered resolvent $P_{\overline{\mathcal{S}}}(A_B - \lambda)^{-1}P_{\overline{\mathcal{S}}}$ jumps at $k \in \mathbb{R}$ iff $k \notin I'$, i.e. the location of the jumps of the bordered resolvent coincides with the jumps of M_B . (Compare to Theorem 6.5, where we can only border the resolvent by finite dimensional projections.)

8.1.2. *Results when ϕ and ψ are not disjointly supported.* We consider a more general case when ϕ and ψ do not necessarily have disjoint supports.

Theorem 8.11. (1) Let $\Omega^c = \{x : \phi(x) \neq 0\} \cup \{x : \psi(x) \neq 0\}$. Then $g \in \mathcal{S}^\perp$ implies $\{x : g(x) \neq 0\} \subseteq \Omega^c$ (up to a set of measure zero). In particular, $\mathcal{S}^\perp \subseteq L^2(\Omega^c)$ and $\overline{\mathcal{S}} \supseteq L^2(\Omega)$.
 (2) Consider $\psi = \alpha\chi_I$ for some constant α and a set I of finite measure and assume $\phi|_{I^c} = 0$ a.e. Then $\overline{\mathcal{S}} = L^2(\mathbb{R})$.

We finish this subsection by showing that in our situation we can improve on Theorems 4.1, 4.6 and 7.1 by recovering the M -function from one bordered resolvent.

Theorem 8.12. As before, let $\Omega_\psi = \{x \in \mathbb{R} : \psi(x) \neq 0\}$ and $\Omega_\phi = \{x \in \mathbb{R} : \phi(x) \neq 0\}$. Let $\Omega = \mathbb{R} \setminus (\Omega_\phi \cup \Omega_\psi)$ and assume that we know a set of non-zero measure $\Omega' \subseteq \Omega$. Then the M -function can be recovered from one bordered resolvent.

Remark 8.13. The converse is not possible. The asymptotics of the M -function at $i\infty$ allow us to recover B and thus $\widehat{\psi}(\lambda)\widehat{\phi}(\lambda)$ for any λ . However, only knowing the product for example makes it impossible to distinguish the expression for A from the operator expression obtained by replacing ψ by $\overline{\phi}$ and ϕ by $\overline{\psi}$, respectively.

8.2. **Results with $\phi, \psi \in H_2^+$.** We note that in the Fourier picture described in Remark 3.6, the condition that $\phi, \psi \in H_2^+$ corresponds to $\mathcal{F}\phi, \mathcal{F}\psi$ being supported in \mathbb{R}^- (by the Paley-Wiener Theorem [28]). A similar remark applies to the next subsection when $\overline{\phi}, \psi \in H_2^+$ where the Fourier transforms will be supported on different half lines. Moreover, similar results will hold if both $\phi, \psi \in H_2^-$.

Theorem 8.14. Let $\phi, \psi \in H_2^+$. Then

$$\overline{\mathcal{S}} = \overline{\bigvee_{\mu \in \mathbb{C}^+} \frac{1}{x - \mu} + \bigvee_{\mu \in \mathbb{C}^-} \frac{D(\mu) + 2\pi i \overline{\phi}(\mu)\psi(x)}{x - \mu}}.$$

Moreover, if

$$(82) \quad \psi(x) = \sum_{j=1}^N \frac{c_j}{x - z_j}$$

with $c_j \neq 0$, $\Im z_j < 0$ and $z_i \neq z_j$ for $i \neq j$, then

$$\text{def}(\mathcal{S}) = N - P - M - M_0,$$

where $P = \sum p_k$ and p_k is the order of poles of $\overline{\phi(\overline{\mu})}/D_+(\mu)$ in $\mathbb{C}_- \setminus \{z_j\}_{j=1}^N$, $M = \sum m_i$, where m_i are the ‘order of the poles’ of $\overline{\phi(x)}/D_+(x)$ in \mathbb{R} (i.e. m_i is the minimum integer such that

$\overline{\phi(x)}/D_+(x)(x-x_i)^{m_i}$ is square integrable), M_0 corresponds to a degenerated case and is given by

$$M_0 = \left| \{j : \overline{\phi(z_j)} = 0 \text{ and } \lim_{\mu \rightarrow z_j} \frac{2\pi i \bar{\phi}(\mu) c_j}{D_+(\mu)(\mu - z_j)} \neq 1\} \right|$$

and

$$D_+(\mu) := 1 + 2\pi i \sum_{j=1}^N \frac{c_j \overline{\phi(z_j)}}{\mu - z_j}$$

for $\mu \in \mathbb{C}$ is a meromorphic continuation of the rational function $D(\mu)$, $\mu \in \mathbb{C}_+$ to the lower half plane. (Note that this will not coincide with $D(\mu)$ in the lower half plane and that for generic $\phi \in H_2^+$ the continuation of D_- to \mathbb{C}_+ will not even exist).

Proposition 8.15. *Any values can be realized for the defect numbers of \mathcal{S} and $\tilde{\mathcal{S}}$ by suitably choosing rational ϕ and ψ in $H_2^+(\mathbb{R})$.*

We conclude this part with an example.

Example 8.16. *Let*

$$\psi(x) = \frac{\alpha}{x - z_1} \text{ with } z_1 \in \mathbb{C}_-, \alpha \in \mathbb{C} \setminus \{0\} \quad \text{and} \quad \bar{\phi}(x) = \frac{1}{x - w_1} \text{ with } w_1 \in \mathbb{C}_+.$$

The root of $D(\lambda)$ in \mathbb{C}_+ or its analytic continuation $D_+(\lambda)$ in \mathbb{C}_- is $\lambda_0 = z_1 + \frac{2\pi i \alpha}{w_1 - z_1}$. We have three cases for N, P, M, M_0 as in Theorem 8.14:

- (1) If $\lambda_0 \in \mathbb{C}_+$ then $N = 1, P = M = M_0 = 0$,
- (2) if $\lambda_0 \in \mathbb{C}_-$ then $N = P = 1, M = M_0 = 0$,
- (3) if $\lambda_0 \in \mathbb{R}$ then $N = M = 1, P = M_0 = 0$.

Therefore, \mathcal{S}^\perp is non-trivial if and only if $\lambda_0 = z_1 + \frac{2\pi i \alpha}{w_1 - z_1} \in \mathbb{C}_+$. In this case, \mathcal{S}^\perp is one dimensional. Moreover,

$$(83) \quad \mathcal{S}^\perp = \left\{ \frac{\text{const}}{(t - \bar{w}_1)(t - \bar{z}_1 + \frac{2\pi i \bar{\alpha}}{\bar{w}_1 - \bar{z}_1})} \right\} \quad \text{and} \quad \bar{\mathcal{S}} = \{f \in L^2(\mathbb{R}) : (P_+ f)(w_1) = (P_+ f)(\lambda_0)\}.$$

Similarly, $\tilde{\mathcal{S}}^\perp$ is non-trivial if and only if $\tilde{\lambda}_0 := w_1 + \frac{2\pi i \alpha}{w_1 - z_1} \in \mathbb{C}_-$ (and therefore $D(\tilde{\lambda}_0) = 0$). Note that if $\lambda_0 \in \mathbb{C}_+$, then also $\tilde{\lambda}_0 \in \mathbb{C}_+$, whilst if $\tilde{\lambda}_0 \in \mathbb{C}_-$, then also $\lambda_0 \in \mathbb{C}_-$. Therefore, at least one of $\bar{\mathcal{S}}$ and $\tilde{\mathcal{S}}$ is the whole space.

Moreover, we see that the bordered resolvent does not detect the singularities at the eigenvalues $\lambda_0 \in \mathbb{C}_+$ or $\tilde{\lambda}_0 \in \mathbb{C}_-$:

For $\lambda \approx \lambda_0 \in \mathbb{C}_+$ we have from (35) and (36) that

$$(84) \quad (A_B - \lambda)^{-1} = \text{regular part at } \lambda_0 + \frac{\mathcal{P}_{\lambda_0}}{\lambda - \lambda_0},$$

with the Riesz projection \mathcal{P}_{λ_0} given by

$$(85) \quad \mathcal{P}_{\lambda_0} = \langle \cdot, u_1 \rangle u_2,$$

where

$$(86) \quad u_1 = \frac{\phi}{x - \bar{\lambda}_0} \quad \text{and} \quad u_2 = \alpha(z_1 - \lambda_0) \frac{\psi(x) - 2\pi i M_B(\lambda_0) \psi(\lambda_0)}{x - \lambda_0}.$$

Since $u_1 \in \mathcal{S}^\perp$, the singularity is cancelled by $P_{\bar{\mathcal{S}}}$. u_2 is the eigenvector of $A_B - \lambda_0$ (see [27]).

For $\lambda \approx \tilde{\lambda}_0 \in \mathbb{C}_-$ we have again from (35) and (36) that

$$(87) \quad (A_B - \lambda)^{-1} = \text{regular part at } \lambda_0 + \frac{\mathcal{P}_{\tilde{\lambda}_0}}{\lambda - \tilde{\lambda}_0},$$

with

$$(88) \quad \mathcal{P}_{\tilde{\lambda}_0} = \left\langle \cdot, (\tilde{\lambda}_0 - \bar{w}_1) \frac{\phi(x) - 2\pi i \overline{M_B(\tilde{\lambda}_0)} \bar{\phi}(\tilde{\lambda}_0)}{x - \tilde{\lambda}_0} \right\rangle \frac{\alpha\psi}{x - \tilde{\lambda}_0},$$

where $\frac{\psi}{x - \tilde{\lambda}_0}$ is an eigenvector of A_B for all (!) B and lies in $\tilde{\mathcal{S}}^\perp$, so the singularity of the resolvent is cancelled by $P_{\tilde{\mathcal{S}}}$.

We note that this behaviour of the bordered resolvent is in accordance with Theorem 3.6 in [13].

Remark 8.17. We note that in the case when $\phi, \psi \in H_2^+$ taking $\lambda, \mu \in \mathbb{C}_+$, the M -function and the ranges of the solution operators $S_{\lambda, B}$ and \tilde{S}_{μ, B^*} do not depend on ϕ and ψ (see (34) and (32)). In fact, $M_B(\lambda) = [\text{sign}(\Im \lambda)\pi i - B]^{-1}$, $S_{\lambda, B}f = (\Gamma_2 f)(x - \lambda)^{-1}$ and $\tilde{S}_{\mu, B^*}f = (\tilde{\Gamma}_2 f)(x - \mu)^{-1}$. In this highly degenerated case, only the boundary condition B can be obtained. Therefore, in this case a Borg-type theorem allowing recovery of the bordered resolvent from the M -function is not possible, even with knowledge of the ranges of the solution operators in the whole of the suitable half-planes. On the other hand, knowledge of the ranges of the solution operators in both half-planes, together with the M -function at one point allows reconstruction by Theorem 4.8.

8.3. Analysis for the case $\bar{\phi}, \psi \in H_2^+$.

Theorem 8.18. Let $\bar{\phi}, \psi \in H_2^+$. Then if $B \neq -\pi i$, we have $\text{def}(\mathcal{S}_B) = 0$. Similar results hold for $\tilde{\mathcal{S}}_B$ by taking adjoints.

Remark 8.19. We note that the space $\bar{\mathcal{S}}_B$ as defined in (9) can depend on the boundary condition. In the degenerate case $B = -\pi i$ we have

$$\bar{\mathcal{S}}_B = \bigvee_{\mu \in \mathbb{C}^+} \left(\frac{D(\mu) - 2\pi i \bar{\phi}(\mu)\psi(x)}{x - \mu} \right).$$

If ϕ or ψ additionally lies in L^∞ , then this gives $\text{def}(\mathcal{S}_B) = +\infty$. However, we consider this choice of B as a degenerate case, since the hypotheses of Proposition 2.8 are not satisfied.

8.4. The general case $\psi, \phi \in L^2$. We conclude this section by studying the general case. Without assumptions on the support, or the Hardy class of ϕ and ψ , the results are rather complicated. Therefore, in what follows we will not worry about imposing slightly stronger regularity conditions on ϕ and ψ . We first define the following set

$$(89) \quad \begin{aligned} E_0 &:= \{ \alpha \in \mathbb{C} : \exists \text{ a set of positive measure } E \subseteq \mathbb{R} \\ &\text{s.t. } 1 + 2\pi i \alpha (P_+(\bar{\phi}\psi) - \psi(P_+(\bar{\phi}))) = 0 \text{ on } E \}. \end{aligned}$$

Note that E_0 consists of those α such that the factor $[D_+ - 2\pi i (P_+\bar{\phi})\psi]$ appearing in (72) - (74) vanishes on some non-null set E when ψ is replaced by $\alpha\psi$.

Theorem 8.20. Assume (78). Let $\alpha \in E_0$, then $\text{def } \mathcal{S}_\alpha = +\infty$.

Remark 8.21. When considering the corresponding $\tilde{\mathcal{S}}_\alpha$ note that the set

$$\tilde{E}_0 := \{ \alpha : 1 + 2\pi i \bar{\alpha} (P_+(\bar{\psi}\phi) - \phi(P_+(\bar{\psi}))) = 0 \text{ on a set of positive measure} \}$$

does not need to coincide with E_0 , so it is possible to have $\text{def } \mathcal{S}_\alpha \neq \text{def } \tilde{\mathcal{S}}_\alpha$ even for $\alpha \in E_0$.

This happens in Example 8.7, where (if ψ is multiplied by a suitable constant) $\text{def } \mathcal{S}_\alpha = \infty$, while $\text{def } \tilde{\mathcal{S}}_\alpha = 0$.

Theorem 8.22. *Let $\phi \in L^2 \cap L^\infty$ and $\psi \in L^2 \cap C_0(\mathbb{R})$, where $C_0(\mathbb{R})$ is the space of continuous functions vanishing at infinity, and assume $\alpha \notin E_0$.*

(1) *Then $\text{def } \mathcal{S}_\alpha > 0$ iff $(2\pi i\alpha)^{-1} \in \sigma_p(\mathcal{M} + \mathcal{K})$, where*

$$(90) \quad \mathcal{M} = ((P_+ \bar{\phi})\psi - P_+(\psi \bar{\phi}))$$

is a possibly unbounded multiplication operator and

$$(91) \quad \mathcal{K} = \bar{\phi} [P_+ \psi P_- - P_- \psi P_+]$$

is the difference of two compact Hankel operators multiplied by $\bar{\phi}$. Note that $D(\mathcal{M} + \mathcal{K}) = D(\mathcal{M})$, where $D(\mathcal{M})$ is the canonical domain of the multiplication operator.

Moreover,

$$\mathcal{S}_\alpha^\perp = \ker \left(\mathcal{M} + \mathcal{K} - \frac{1}{2\pi i\alpha} \right),$$

so

$$\text{def } \mathcal{S}_\alpha = \dim \ker \left(\mathcal{M} + \mathcal{K} - \frac{1}{2\pi i\alpha} \right).$$

If $(2\pi i\alpha)^{-1} \notin \overline{\text{essran}_{k \in \mathbb{R}} \mathcal{M}(k)}$, then

$$\text{def } \mathcal{S}_\alpha = \dim \ker \left(I + \mathcal{K} \left(\mathcal{M} - \frac{1}{2\pi i\alpha} \right)^{-1} \right) < \infty.$$

(2) *Additionally assume $\mathcal{M}(k)$ is continuous. Then $\mathbb{C} \setminus \overline{\text{Ran } \mathcal{M}(k)}$ is a countable union of disjoint connected domains. Set $\mu = (2\pi i\alpha)^{-1}$. Then in each of these domains we have either*

(a) *$\text{def } \mathcal{S}_\alpha = 0$ whenever μ is in this domain except (possibly) a discrete set, or*

(b) *$\text{def } \mathcal{S}_\alpha \neq 0$ is finite and constant for any μ in the domain except (possibly) a discrete set.*

Moreover, for μ sufficiently large, we have $\text{def } \mathcal{S}_\alpha = 0$.

Although this theorem gives a description of $\bar{\mathcal{S}}_\alpha$ for a rather general case of ψ and ϕ , for concrete examples as investigated in previous subsections it is useful to determine the space explicitly rather than just give the description in terms of operators \mathcal{K} and \mathcal{M} . However, this theorem shows the topological properties of the function $\text{def } \mathcal{S}_\alpha$ in the α -plane.

Example 8.23. *Let*

$$\psi(x) = \alpha \left(\frac{c_1}{x - z_1} + \frac{c_2}{x - z_2} \right) \quad \text{with } z_1 \neq z_2 \in \mathbb{C}_-, \alpha \in \mathbb{C} \setminus \{0\}$$

and

$$\bar{\phi}(x) = \frac{1}{x - w_1} \quad \text{with } w_1 \in \mathbb{C}_+.$$

We wish to analyse the defect as a function of α . By Theorem 8.14, we need to determine the number of roots of the analytic continuation $D_+(\lambda)$ of $D(\lambda)$ in \mathbb{C}_- . Now,

$$(92) \quad D_+(\lambda) = 1 - 2\pi i\alpha \left(\frac{c_1}{(z_1 - w_1)(z_1 - \lambda)} + \frac{c_2}{(z_2 - w_1)(z_2 - \lambda)} \right).$$

After setting $\hat{\mu} := \frac{2\pi i\alpha}{(z_1 - w_1)(z_2 - w_1)}$ a short calculation shows that the roots of $D_+(\lambda)$ solve

$$(93) \quad \lambda^2 + \lambda(d_1 \hat{\mu} - z_1 - z_2) + d_2 \hat{\mu} + z_1 z_2 = 0,$$

where

$$d_1 = c_1(z_2 - w_1) + c_2(z_1 - w_1) \quad \text{and} \quad d_2 = -c_1 z_2(z_2 - w_1) - c_2 z_1(z_1 - w_1).$$

In particular, for $\hat{\mu} = 0$ the roots are $z_1, z_2 \in \mathbb{C}_-$. By continuity, for small $|\alpha|$, by Theorem 8.14 we have $\text{def } \bar{\mathcal{S}}_\alpha = 0$.

For a polynomial $\lambda^2 + p\lambda + q = 0$, an elementary calculation shows that it has a real root iff

$$(94) \quad (\Im q)^2 = (\Im p)(\Re p \Im q - \Re q \Im p) \quad \text{and} \quad 4\Re q \leq |p|^2.$$

We now analyse the defect in a few examples.

(A) We first make the specific choice

$$\psi(x) = \alpha \left(\frac{-2}{x+i} + \frac{3}{x+2i} \right) \quad \text{and} \quad \bar{\phi}(x) = \frac{1}{x-i}.$$

Then $d_1 = 0, d_2 = -6$ and the equation in (94) becomes

$$(95) \quad (\Im \hat{\mu})^2 = \frac{1}{2}(1 + 3\Re \hat{\mu}).$$

All $\hat{\mu}$ satisfying (95) satisfy the inequality in (94). This gives a parabola in the α -plane (or equivalently the $\hat{\mu}$ -plane) with $\text{def } \bar{\mathcal{S}}_\alpha = 0$ inside or on the parabola and $\text{def } \bar{\mathcal{S}}_\alpha = 1$ outside. In the $1/\alpha$ -plane this gives a curve whose interior is a petal-like shape with $\text{def } \bar{\mathcal{S}}_\alpha = 0$ for $1/\alpha$ outside or on the curve and $\text{def } \bar{\mathcal{S}}_\alpha = 1$ for $1/\alpha$ inside the curve.

(B) We now return to the formula for D_+ in (92). Setting $\mu = (2\pi i \alpha)^{-1}$, we have

$$(96) \quad \mu = \frac{c_1}{(z_1 - w_1)(z_1 - \lambda)} + \frac{c_2}{(z_2 - w_1)(z_2 - \lambda)}.$$

Clearly for $\lambda \rightarrow \pm\infty$, we have that $\mu = 0$. We now choose c_1, c_2 to get another real root at $\lambda = 0$. Consider

$$\psi(x) = \alpha \left(\frac{-1}{x+i} + \frac{3}{x+2i} \right) \quad \text{and} \quad \bar{\phi}(x) = \frac{1}{x-i}.$$

In the μ -plane this leads to one petal. As λ runs through \mathbb{R} , this curve is covered twice (once for $\lambda < 0$ and once for $\lambda > 0$). We have $\text{def } \bar{\mathcal{S}}_\alpha = 0$ for μ outside the curve and $\text{def } \bar{\mathcal{S}}_\alpha = 2$ for μ inside the curve. On the curve we have $\text{def } \bar{\mathcal{S}}_\alpha = 0$. The double covering of the curve allows the jump of 2 in the defect when crossing the curve.

(C) More generally, if ψ has N terms, then the problem of finding real roots of $D_+(\lambda)$ leads to studying the real zeroes of

$$\xi(\lambda) := \sum_{k=1}^N \frac{a_k}{z_k - \lambda}, \quad \text{where} \quad a_k = c_k \bar{\phi}(z_k).$$

Generically ξ will not have real zeroes and we will only get one petal in the μ -plane. However, we can arrange it that ξ has $N - 1$ real zeroes which leads to N petals in the μ -plane. Assume $a_N \neq 0$. Then to do this, we need to solve the linear system,

$$(97) \quad Z \begin{pmatrix} a_1 \\ \vdots \\ a_{N-1} \end{pmatrix} = \begin{pmatrix} -\frac{a_N}{z_N - \lambda_1} \\ \vdots \\ -\frac{a_N}{z_N - \lambda_{N-1}} \end{pmatrix},$$

where the matrix Z has jk -component given by $z_{jk} = (z_k - \lambda_j)^{-1}$. Z is invertible whenever all $z_k \in \mathbb{C}_-, \lambda_j \in \mathbb{R}$ are distinct.

For the example in Figure 1, the defect in each of the components is given by $4 - \nu_-$ where ν_- denotes the number of roots of D_+ in \mathbb{C}_- (by Theorem 8.14). At each curve precisely one of the roots crosses from the lower to the upper half-plane, thus increasing the defect by 1. On the curve itself, one root is on the real axis and by Theorem 8.14, the defect coincides with the smaller of the defects on the components on each side of the

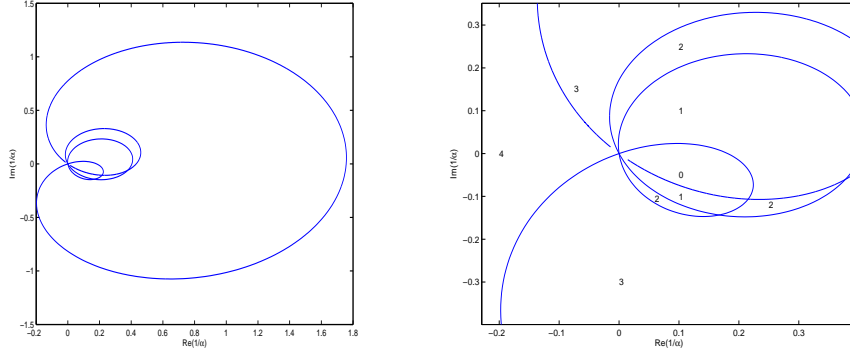


FIGURE 1. The curve in the $1/\alpha$ -plane along which D_+ has a real root for the case $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -2$, $z_1 = -i$, $z_2 = 1 - i$, $z_3 = -2 - i$, $z_4 = 3 - 2i$ and $a_4 = 1$. On the right, zoom of part of the curve including the number of roots of D_+ in \mathbb{C}_- in different components.

curve. By a similar reasoning at the three non-zero points of self-intersection of the curve the defect coincides with the smallest defect of the neighbouring components.

This example displays the analytical nature of finding the defect in terms of the location of roots of D_+ using Theorem 8.14. On the other hand, it also displays the topological nature of the same situation mentioned in Theorem 8.22. The complex $1/\alpha$ -plane is separated into components in which the defect is constant everywhere (in this example the exceptional discrete set is empty). The curves are the range of $2\pi i\mathcal{M}(t)$ on the real axis.

9. SPECTRA OF TOEPLITZ OPERATORS AND DETECTABILITY

In Section 8 Hankel operators played a special role in the analysis (see e.g. Theorem 8.22). However, for another class of examples of the Friedrichs model, the theory of Toeplitz operators is the main instrument of our analysis. We will discuss this type of examples and the related detectability problem here. The proofs and some auxiliary results can be found in Appendix B.

We will study the operator $T := \bar{\phi}P_+\psi$ acting on L^2 . It is closely related to the Toeplitz operator $T_a : H_2^+ \rightarrow H_2^+$ given by $T_a u = P_+ a u = P_+ a P_+ u$, where $a = \psi\bar{\phi}$.

Assumptions 9.1. For this section, we will make the following assumptions on the functions $\phi, \psi \in L^2$: (i) $a(z) := \psi(z)\bar{\phi}(z) \in H_1^- \setminus \{0\}$, (ii) $\phi \in H_2^-$ and (iii) $\phi, \psi \in L^\infty$.

Remark 9.2. (1) Under the above assumptions, we have $D_+(\lambda) \equiv 1$ and $P_+\bar{\phi} = \bar{\phi}$, $P_-\bar{\phi} = 0$.
 (2) Choosing $\phi = (x - i)^{-\frac{1}{2}-\varepsilon}$ for some $\varepsilon > 0$ and a suitable choice of the branch cut, we have $\phi \in H_2^-$. To satisfy the first assumption, we then require $\psi(x) = a(x)(x + i)^{\frac{1}{2}+\varepsilon} \in L^2$, or $a \in L^2(\mathbb{R}; (1 + x^2)^{\frac{1}{2}+\varepsilon})$. Therefore, the assumption that $\phi \in H_2^-$ only imposes a mild extra condition on the decay of a at infinity.
 (3) The third assumption is only introduced for the sake of convenience and may be significantly relaxed. However, this would introduce more technical details which would obscure the main results. It means that the operator T is a bounded operator in L^2 and T_a in H_2^+ , in particular, they are defined on the whole space.

In the next theorem we will make use of the canonical Riesz-Nevanlinna factorization theorem (see [28, Pages 199-200]). For the reader's convenience, we state it here for functions in the lower half plane: If $f \not\equiv 0$, $f \in H_p^-$ for $p \geq 1$, then up to constant multiples, f can be factorized uniquely as

$$(98) \quad f(z) = B(z)\Sigma(z)G(z),$$

where

- $B(z)$ is a Blaschke product with

$$B(z) = \prod_k \left(e^{i\theta_k} \frac{z - z_k}{z - \bar{z}_k} \right) \text{ for } \Im z < 0,$$

where z_k are all the roots of f in \mathbb{C}_- and the real θ_k are chosen so that

$$e^{i\theta_k} \frac{i - z_k}{i - \bar{z}_k} \geq 0;$$

- $\Sigma(z)$ is the singular factor given by

$$\Sigma(z) = \exp \left(\frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{i}{z - t} + \frac{it}{t^2 + 1} \right) d\sigma(t) \right)$$

with $d\sigma(t) \geq 0$ a singular measure (w.r.t. Lebesgue measure) such that $\int_{-\infty}^{\infty} \frac{d\sigma(t)}{t^2 + 1} < \infty$;

- $G(z)$ is the outer factor given by

$$G(z) = \exp \left(-\frac{i}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{z - t} + \frac{t}{t^2 + 1} \right) \log |f(t)| dt \right).$$

Theorem 9.3. Define the operators T on L^2 and $T_a : H_2^+ \rightarrow H_2^+$ as above and let $\mu_\alpha = (2\pi i \alpha)^{-1}$. Then

- (1) $\text{def}(\mathcal{S}_\alpha) = \dim \ker (T_a - \mu_\alpha)$. Moreover,

$$\mathcal{S}_\alpha^\perp = (\ker (T - \mu_\alpha))^* = \overline{\psi^{-1}}[(\mu_\alpha P_+ + P_- a P_+) \ker (T_a - \mu_\alpha)]^*.$$

Here, we use $(\ker (T - \mu_\alpha))^*$ to denote the set of complex conjugates of functions in $\ker (T - \mu_\alpha)$. This will be used to distinguish it from the closure in cases where the meaning may be ambiguous.

- (2) We have $\mathcal{S}_\alpha^\perp = \phi(H_2^- \ominus (a(z) - \mu_\alpha)H_2^-)$. In particular, $\phi(H_2^- \ominus (a(z) - \mu_\alpha)H_2^-)$ is a closed subspace.
- (3) Consider the canonical factorisation in \mathbb{C}_- of $a(z) - \mu_\alpha = B_\alpha(z)\Sigma_\alpha(z)G_\alpha(z)$ for a fixed $\alpha \in \mathbb{C}$ where B_α is a Blaschke product containing all zeros of $a(z) - \mu_\alpha$ in \mathbb{C}_- , Σ_α is the singular part and G_α is an outer function. Then

$$\overline{(a - \mu_\alpha)H_2^-} = \overline{B_\alpha(z)\Sigma_\alpha(z)H_2^-} \equiv B_\alpha(z)\Sigma_\alpha(z)H_2^-.$$

and

$$(99) \quad \text{def}(\mathcal{S}_\alpha) = \begin{cases} \infty & \text{if } \Sigma_\alpha \not\equiv 1, \\ \text{number of roots of } B_\alpha(z) \text{ counted with multiplicity} & \text{if } \Sigma_\alpha \equiv 1. \end{cases}$$

Example 9.4. For fixed $\alpha \in \mathbb{C}$, we consider a canonical decomposition of $(a - \mu_\alpha)$ in \mathbb{C}_- of the following form: Choose any B_a with zeroes in a finite box and the singular measure in Σ_a with bounded support. Then $B_a \Sigma_a \rightarrow 1$ at infinity. Choose

$$G_a(z) = -\mu_\alpha \frac{(z - a_1)(z + \sigma)}{z^2 + \tau z + \rho},$$

where $a_1 \in \mathbb{C}_+$, $\sigma \in \mathbb{C}_-$, $\tau \in \mathbb{C}_-$ and $\rho \ll -1$.

Being contractive in \mathbb{C}_- the product $B_a \Sigma_a$ behaves like $e^{\frac{b}{z}}$ with $b \in \mathbb{C}_+$ at ∞ . We can choose the constants above such that $a_1 - \sigma = b - \tau \in \mathbb{C}_+$. Therefore $a(z) = O(1/z^2)$ at ∞ . Defining $\phi(z) = (z - i)^{-1} \in H_2^-$, we have $\phi \sim 1/z$ at ∞ , so $\psi := a/\bar{\phi} = O(1/z)$ belongs to L^2 .

Choosing ρ sufficiently negative, we get that both roots of $z^2 + \tau z + \rho$, approximately $-\tau/2 \pm \sqrt{-\rho}$, lie in \mathbb{C}_+ . This gives $G_a \in H_\infty^-$ and since both its roots a_1 and $-\sigma$ lie in \mathbb{C}_+ , G_a is outer in \mathbb{C}_-

By Theorem 9.3, the number of roots in \mathbb{C}_- equals $\text{def } \mathcal{S}$ provided $\Sigma_a \equiv 1$ and therefore all natural numbers are possible as indices.

APPENDIX A. PROOFS OF THE RESULTS OF SECTION 8

We start with an elementary uniqueness lemma, whose proof we include for completeness.

Lemma A.1. *Let*

$$G(\mu) = \begin{cases} G_+(\mu), & \mu \in \mathbb{C}_+, \\ G_-(\mu), & \mu \in \mathbb{C}_-. \end{cases}$$

with $G_\pm \in H_1^\pm + H_2^\pm := \{G_1 + G_2 : G_1 \in H_1^\pm, G_2 \in H_2^\pm\}$. Then the jump across the real axis $[G] \equiv 0$ iff and only iff $G \equiv 0$.

Proof. We show that if the jump vanishes, then G vanishes, the other implication is trivial.

By $G(x \pm i0)$ we denote the boundary values of the functions G_\pm which exist a.e. on \mathbb{R} (see [28]). Since $[G] \equiv 0$, we have $G(x) := G(x + i0) = G(x - i0)$. Now, $G(x + i0) \in H_1^+ + H_2^+$, so for all $\lambda \in \mathbb{C}_-$ we have

$$\int_{\mathbb{R}} \frac{G(t)}{t - \lambda} \equiv 0.$$

Similarly, $G(x - i0) \in H_1^- + H_2^-$, so for all $\lambda \in \mathbb{C}_+$ we have

$$\int_{\mathbb{R}} \frac{G(t)}{t - \lambda} \equiv 0.$$

Combining this gives

$$\int_{\mathbb{R}} \frac{G(t)}{t - \lambda} \equiv 0 \quad \text{for all } \lambda \notin \mathbb{R}.$$

We want to show that this implies $G(x) = 0$ for almost every $x \in \mathbb{R}$. Clearly, taking complex conjugates,

$$\int_{\mathbb{R}} \frac{\Re G(t)}{t - \lambda} \equiv 0 \equiv \int_{\mathbb{R}} \frac{\Im G(t)}{t - \lambda},$$

so we may now assume without loss of generality that G is real. Then

$$0 = \Im \int_{\mathbb{R}} \frac{G(t)}{t - \lambda} = \Im \lambda \int_{\mathbb{R}} \frac{G(t)}{|t - \lambda|^2} = \pi P_\varepsilon(G)(k),$$

where $\lambda = k + i\varepsilon$ and P_ε denotes the Poisson transformation.

Now, let I be any bounded open interval in \mathbb{R} and $k \in I$. As $G = G_1 + G_2 \in L^1(\mathbb{R}) + L^2(\mathbb{R})$, we have $\chi_I G \in L^1(\mathbb{R})$ and

$$\varepsilon \int_{I^c} \frac{G_1(x) + G_2(x)}{|x - \lambda|^2} dx \leq \varepsilon \int_{I^c} \frac{|G_1(x)| + |G_2(x)|}{|x - k|^2} dx = o(1),$$

where we have used that $G_2(x)/|x - k|^2 \in L^1(I^c)$. Therefore,

$$P_\varepsilon(G) = P_\varepsilon(\chi_I G + \chi_{I^c} G) = P_\varepsilon(\chi_I G) + o(1) \rightarrow \chi_I G \text{ as } \varepsilon \rightarrow 0.$$

This shows that $\chi_I G(x) = 0$ and as I was arbitrary, we get $G(x) = 0$ for almost every $x \in \mathbb{R}$. As the boundary values of G_\pm vanish identically, we have $G_\pm \equiv 0$. \square

We can now prove the characterisation of \mathcal{S}^\perp .

Proof of Proposition 8.1. (1) Noting the form of elements from $\text{Ran } S_{\lambda,B}$ from (32), we get that

$$\mathcal{S}^\perp = \left\{ g \in L^2(\mathbb{R}) : \text{for all } \mu \notin \mathbb{R} \right. \\ \left. F_\pm(\mu) := \left\langle \frac{1}{x-\mu}, g \right\rangle - \frac{1}{D(\mu)} \left\langle \frac{1}{x-\mu}, \phi \right\rangle \left\langle \frac{1}{x-\mu} \psi, g \right\rangle = 0 \right\}.$$

Here the index \pm indicates which half plane μ lies in. Then

$$(100) \quad F_\pm(\mu) = \widehat{g}(\mu) - \frac{1}{D(\mu)} \widehat{\phi}(\mu) \widehat{\psi g}(\mu).$$

First assume $g \in \mathcal{S}^\perp$. Let $\mu \in \mathbb{C}_+ \setminus \{D(\mu) = 0\}$. Then $F_+(\mu) = 0$ is equivalent to

$$(101) \quad (P_+ \bar{g})(\mu) = \frac{2\pi i}{D_+(\mu)} (P_+ \bar{\phi})(\mu) P_+(\psi \bar{g})(\mu).$$

Similarly, if $\mu \in \mathbb{C}_- \setminus \{D(\mu) = 0\}$, then $F_-(\mu) = 0$ is equivalent to

$$(102) \quad (P_- \bar{g})(\mu) = -\frac{2\pi i}{D_-(\mu)} (P_- \bar{\phi})(\mu) P_-(\psi \bar{g})(\mu).$$

This proves the first implication in the statement.

Now, as the zeroes of $D(\mu)$ in \mathbb{C}_\pm form a discrete set, the right hand sides of (101) and (102) must lie in H_2^+ and H_2^- , respectively, showing (i) and (ii). Since $\bar{g} = P_+ \bar{g} + P_- \bar{g}$, we also get (iii). For the reverse implications, simply apply P_+ and P_- to \bar{g} as given in (iii).

(2) We first show the equivalence in (72). Let $g \in \mathcal{S}^\perp$ and apply P_\pm to (71), part (iii), keeping (71) parts (i) and (ii) in mind. Then

$$P_+ \bar{g} - \frac{2\pi i}{D_+} (P_+ \bar{\phi}) P_+(\psi \bar{g}) = 0 \quad \text{and} \quad P_- \bar{g} + \frac{2\pi i}{D_-} (P_- \bar{\phi}) P_-(\psi \bar{g}) = 0.$$

Since D_\pm only have discrete zeroes, this is equivalent to

$$D_\pm P_\pm(\bar{g}) \mp 2\pi i (P_\pm \bar{\phi}) P_\pm(\psi \bar{g}) = 0.$$

In particular, on \mathbb{R} we have

$$D_+ P_+(\bar{g}) - 2\pi i (P_+ \bar{\phi}) P_+(\psi \bar{g}) = -D_- P_-(\bar{g}) - 2\pi i (P_- \bar{\phi}) P_-(\psi \bar{g}) \quad (a.e.).$$

Since for the boundary values on \mathbb{R} we have $D_- = 1 - 2\pi i P_-(\psi \bar{\phi}) = D_+ - 2\pi i \psi \bar{\phi}$ a.e., this is equivalent to

$$D_+(P_+(\bar{g}) + P_-(\bar{g})) - 2\pi i (P_+ \bar{\phi}) P_+(\psi \bar{g}) = 2\pi i \psi \bar{\phi} P_-(\bar{g}) - 2\pi i (P_- \bar{\phi}) P_-(\psi \bar{g}) \quad (a.e.)$$

which can be rewritten as

$$D_+ \bar{g} - 2\pi i (P_+ \bar{\phi}) \psi \bar{g} = 2\pi i \psi \bar{\phi} P_-(\bar{g}) - 2\pi i \bar{\phi} P_-(\psi \bar{g}) \quad (a.e.),$$

giving the right hand side of (72).

On the other hand, assume the right hand side of (72) holds. Retracing our steps above, this gives

$$(103) \quad \widetilde{F}_+ := D_+ P_+(\bar{g}) - 2\pi i (P_+ \bar{\phi}) P_+(\psi \bar{g}) = -D_- P_-(\bar{g}) - 2\pi i (P_- \bar{\phi}) P_-(\psi \bar{g}) =: \widetilde{F}_- \quad (a.e.).$$

Using Hölder's inequality and boundedness of the Riesz projections $P_\pm : L^p \rightarrow L^p$ for $1 < p < \infty$ for the cases $p = \frac{4}{3}$, $p = 2$ and $p = 4$ (see proof of Proposition A.2 for more details), our conditions on ψ and ϕ guarantee that $\widetilde{F}_\pm \in H_1^\pm + H_2^\pm$. Moreover, (103) states that $[\widetilde{F}] = 0$. By Lemma A.1 we have $\widetilde{F} = 0$ and so

$$D_+ P_+(\bar{g}) - 2\pi i (P_+ \bar{\phi}) P_+(\psi \bar{g}) = 0 \quad \text{and} \quad D_- P_-(\bar{g}) + 2\pi i (P_- \bar{\phi}) P_-(\psi \bar{g}) = 0.$$

Therefore all conditions on the right hand side of (70) are satisfied and $g \in \mathcal{S}^\perp$.

Using the identity $P_- = I - P_+$ then gives (73) and a similarly simple calculation leads to (74). □

A.1. Results depending on the support of ϕ and ψ .

Proposition A.2. *Let either $\phi \in L^2$ and $\psi \in L^2 \cap L^\infty$ or $\phi, \psi \in L^2 \cap L^4$ be such that $\phi\psi = 0$. Then*

$$(104) \quad g \in \overline{\mathcal{S}}^\perp \iff \bar{g} - 2\pi i(P_+\bar{\phi})P_+(\psi\bar{g}) + 2\pi i(P_-\bar{\phi})P_-(\psi\bar{g}) = 0 \text{ (a.e.)}$$

$$(105) \quad \iff \bar{g} - 2\pi i\bar{\phi}P_+(\psi\bar{g}) + 2\pi i(P_-\bar{\phi})\psi\bar{g} = 0 \text{ (a.e.)}$$

Define $L_0 = 2\pi i(\bar{\phi}P_+\psi - (P_-\bar{\phi})\psi) \equiv 2\pi i(\bar{\phi}P_+\psi + (P_+\bar{\phi})\psi)$ on its maximal domain $D(L_0) = \{u \in L^2 : L_0u \in L^2\}$. Then we have $\overline{\mathcal{S}} = L^2(\mathbb{R})$ if and only if 1 is not an eigenvalue of the operator L_0 on $L^2(\mathbb{R})$.

Proof. In this case, $D(\lambda) \equiv 1$, so (104) is equivalent to condition (iii) on the right hand side of (71) and one implication is trivial. By Proposition 8.1 it is sufficient to show that conditions (i) and (ii) on the right hand side of (71) hold. In our case, this simplifies to

$$(i') (P_+\bar{\phi})P_+(\psi\bar{g}) \in H_2^+ \text{ and } (ii') (P_-\bar{\phi})P_-(\psi\bar{g}) \in H_2^-.$$

In the case when $\phi \in L^2$ and $\psi \in L^2 \cap L^\infty$, we have that $P_+\bar{\phi} \in L^2$ and $P_+(\psi\bar{g}) \in L^2$ by boundedness of the Riesz projection $P_+ : L^2 \rightarrow L^2$, so the product lies in H_1^+ (see [28]). On the other hand, if $\phi, \psi \in L^2 \cap L^4$, then $\psi\bar{g} \in L^{4/3}$ by Hölder's inequality, so $P_+(\psi\bar{g}) \in L^{4/3}$. Also $P_+\bar{\phi} \in L^4$, so by Hölder's inequality, we again get that the product lies in H_1^+ . Similarly $(P_-\bar{\phi})P_-(\psi\bar{g}) \in H_1^-$.

From (104), $\bar{g} = 2\pi i(P_+\bar{\phi})P_+(\psi\bar{g}) - 2\pi i(P_-\bar{\phi})P_-(\psi\bar{g})$ which gives a decomposition of $\bar{g} \in L^2$ into its unique H^+ and H^- components, whence we obtain (i') and (ii').

Now,

$$\begin{aligned} \bar{g} &= 2\pi i(P_+\bar{\phi})P_+(\psi\bar{g}) - 2\pi i(P_-\bar{\phi})P_-(\psi\bar{g}) \\ &= 2\pi i[(P_+\bar{\phi})P_+\psi - (P_-\bar{\phi})P_-\psi] \bar{g} \\ &= 2\pi i[\bar{\phi}P_+\psi - (P_-\bar{\phi})(P_+\psi + P_-\psi)] \bar{g} = L_0\bar{g}, \end{aligned}$$

which shows (105) and the statement about L_0 . □

Remark A.3. *Note that $D(L_0)$ is dense in L^2 if $\phi, \psi \in L^2 \cap L^4$ or if $\phi \in L^{2+\varepsilon}$ for some $\varepsilon > 0$ and $\psi \in L^2 \cap L^\infty$, as it contains $L^2 \cap L^\infty$.*

Proof of Theorem 8.5. We recall that $\mathbb{R} = \Omega_\phi \cup \Omega_\psi \cup \Omega$. Without loss of generality, we may assume $\Omega_\phi \cap \Omega_\psi = \emptyset$. Suppose $g \in \overline{\mathcal{S}}^\perp$. Using (105), we then have three cases:

(1) $k \in \Omega_\phi$: we have $0 = \bar{g}(k) - \bar{\phi}(k)\widehat{\psi\chi_{\Omega_\psi}\bar{g}}(k+i0)$. Hence,

$$(106) \quad \chi_{\Omega_\phi}\bar{g}(k) = \bar{\phi}(k)\widehat{\psi\chi_{\Omega_\psi}\bar{g}}(k+i0)$$

and so $g|_{\Omega_\psi}$ completely determines $g|_{\Omega_\phi}$.

(2) $k \in \Omega_\psi$: we have $0 = \bar{g}(k) - \widehat{\bar{\phi}(k-i0)\psi(k)\bar{g}(k)}$. Hence, almost everywhere we have

$$(107) \quad \chi_{\Omega_\psi}\bar{g}(k) \left(1 - \widehat{\bar{\phi}(k-i0)\psi(k)}\right) = 0.$$

(3) $k \in \Omega$: we have

$$(108) \quad g|_\Omega = 0.$$

This gives three necessary and sufficient conditions for g to lie in $\overline{\mathcal{S}}^\perp$.

We now prove the statements (i)-(iii).

- (i) If $\widehat{\phi}(k - i0)\psi(k) \neq 1$ for almost every $k \in \Omega_\psi$, (80) implies that $g|_{\Omega_\psi} = 0$, and so by (79) we have $g|_{\Omega_\phi} = 0$ whence $g \equiv 0$ and $\overline{\mathcal{S}} = L^2(\mathbb{R})$.
- (ii) Choose g on $\Omega_{\psi,0}$ to be an arbitrary non-zero $(L^2 \cap L^\infty)(\Omega_{\psi,0})$ -function (in the case when $\phi \in L^{2+\varepsilon}$, $\psi \in L^\infty$, we may even choose g arbitrary in $(L^2 \cap L^{\frac{2(2+\varepsilon)}{\varepsilon}})(\Omega_{\psi,0})$). Extending g by zero to Ω_ψ and then using Hölder's inequality and boundedness of P_+ , we automatically get that $\overline{\phi} \left(\widehat{\psi \chi_{\Omega_\psi} \bar{g}} \right)_+ \in L^2$. This then determines g on Ω_ψ from (79) and extending g to Ω by 0, we have $g \in \mathcal{S}^\perp$.

Now let $f \in \overline{\mathcal{S}}$ and choose $g \in (L^2 \cap L^\infty)(\Omega_{\psi,0})$. Then

$$\begin{aligned} 0 &= \int_{\mathbb{R}} f \bar{g} = \int_{\Omega_{\psi,0}} f \bar{g} + \int_{\Omega_\phi} f \bar{g} = \int_{\Omega_{\psi,0}} f \bar{g} + \int_{\Omega_\phi} f \overline{\widehat{\phi} \left(\widehat{\psi \bar{g}} \right)_+} \\ &= \int_{\Omega_{\psi,0}} f \bar{g} + \int_{\mathbb{R}} f \overline{\widehat{\phi} \left(\widehat{\psi \bar{g}} \right)_+} = \int_{\Omega_{\psi,0}} f \bar{g} - \int_{\mathbb{R}} \left(\widehat{f \overline{\phi}} \right)_- \psi \bar{g} \\ &= \int_{\Omega_{\psi,0}} f \bar{g} - \int_{\Omega_{\psi,0}} \left(\widehat{f \overline{\phi}} \right)_- \psi \bar{g} = \int_{\Omega_{\psi,0}} \left(f - \psi \left(\widehat{f \overline{\phi}} \right)_- \right) \bar{g}, \end{aligned}$$

from which it follows that $\bar{g} \mapsto \int_{\Omega_{\psi,0}} \left(f - \psi \left(\widehat{f \overline{\phi}} \right)_- \right) \bar{g}$ is a bounded linear functional for all g in the dense set $(L^2 \cap L^\infty)(\Omega_{\psi,0})$. Hence, $\psi \left(\widehat{f \overline{\phi}} \right)_- \in L^2(\Omega_{\psi,0})$ and we see that $f \in \overline{\mathcal{S}}$ must satisfy $f = \psi \left(\widehat{f \overline{\phi}} \right)_-$ on $L^2(\Omega_{\psi,0})$.

- (iii) If ψ and ϕ are both bounded, then choosing $g \in L^2 \cap L^\infty(\Omega_{\psi,0})$ and determining g on Ω_ϕ from (79) will give a dense set of g in \mathcal{S}^\perp , as $g \mapsto \overline{\phi} \left(\widehat{\psi \bar{g}} \right)_+$ is bounded. The calculation in (ii) can then be repeated for a dense set of g in \mathcal{S}^\perp which gives the needed equality of the two sets.

□

Proof of Theorem 8.8. From (105), we have

$$(109) \quad g \in \mathcal{S}^\perp \iff \bar{g}(k) - \overline{\phi}(k) \left(\widehat{\psi \bar{g}} \right)_+ - \left(\widehat{\phi} \right)_- \psi \bar{g} = 0 \text{ a.e.} \iff (1 - \psi \left(\widehat{\phi} \right)_+) (\bar{g} - \overline{\phi} \left(\widehat{\psi \bar{g}} \right)_+) = 0 \text{ a.e.}$$

We have two cases: In the first case, $\bar{g} - \overline{\phi} \left(\widehat{\psi \bar{g}} \right)_+ = 0$ a.e. Then multiplying by ψ and using the condition on the supports, we get $\psi \bar{g} = 0$ and hence $\bar{g} = 0$.

In the second case, $\bar{g} - \overline{\phi} \cdot 2\pi i P_+(\psi \bar{g}) \neq 0$. Then there exists a set E of positive measure such that $\bar{g} - \overline{\phi} \cdot 2\pi i P_+(\psi \bar{g}) \neq 0$ on E and $\left(1 - \psi \left(\widehat{\phi} \right)_+ \right) \Big|_E = 0$ a.e.

We now show that if there exists a set E of positive measure such that $(1 - \psi \left(\widehat{\phi} \right)_+) \Big|_E = 0$ a.e., then $\mathcal{S}^\perp \neq \{0\}$. Note first that $E \subset \Omega_\psi$. Choose any non-zero $\tilde{g} \in (L^2 \cap L^\infty)(E)$, $\tilde{g} \neq 0$ and continue it to \mathbb{R} by zero.

Define

$$\bar{g}(k) = \begin{cases} \overline{\tilde{g}(k)}, & k \in E, \\ 0, & k \in \Omega_\psi \setminus E, \\ \overline{\phi} 2\pi i P_+(\psi \tilde{g}), & k \notin \Omega_\psi. \end{cases}$$

By (109), to show $g \in \mathcal{S}^\perp$ we only require $g \in L^2$. This follows immediately from the condition (78). From the freedom in the choice of \tilde{g} , it is clear that $\text{def } \mathcal{S}$ is infinite. \square

Lemma A.4. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be measurable. Consider the set $E_\alpha \subset \mathbb{R}$ given by $E_\alpha = \{x \in \mathbb{R} : f(x) = \alpha\}$, $\alpha \in \mathbb{C}$. Then $\{\alpha \in \mathbb{C} : |E_\alpha| > 0\}$ is countable.*

Proof. $\{\alpha \in \mathbb{C} : |E_\alpha| > 0\} = \bigcup_{n=1}^{\infty} \{\alpha \in \mathbb{C} : |E_\alpha| > 1/n\}$. Assume the set is not countable. Then there exists ε_0 such that $J := \{\alpha \in \mathbb{C} : |E_\alpha| > \varepsilon_0\}$ is uncountable. Moreover if $\alpha \neq \alpha'$, $E_\alpha \cap E_{\alpha'} = \emptyset$. Now $\mathbb{C} \supset \bigcup_{\alpha \in J} E_\alpha$ is a disjoint union of uncountably many sets of measure greater than ε_0 . Choose $\tilde{f}_\alpha = \chi_{\tilde{E}_\alpha}$ where $\tilde{E}_\alpha \subseteq E_\alpha$ with $|\tilde{E}_\alpha| = \varepsilon_0$. Then $\|\tilde{f}_\alpha\| = \sqrt{\varepsilon_0}$ and $\tilde{f}_\alpha \perp \tilde{f}_{\alpha'}$ for $\alpha \neq \alpha'$. This contradicts separability of $L^2(\mathbb{R}^2)$. Therefore, $\{\alpha \in \mathbb{C} : |E_\alpha| > 0\}$ must be countable. \square

Proof of Theorem 8.9. (1) This is immediate from Theorem 8.8 with the help of Lemma A.4. (2) From (105), we have $g \in \mathcal{S}^\perp$ if and only if

$$(110) \quad \bar{g}(k) - \bar{\phi}(k) \widehat{\psi \bar{g}}(k + i0) - \widehat{\bar{\phi}}(k - i0) \psi(k) \bar{g}(k) = 0.$$

Multiplying by $\psi(k)$, and setting $g_\psi := \psi \bar{g}$ we get

$$(111) \quad g_\psi(k) = \psi(k) \widehat{\bar{\phi}}(k - i0) g_\psi(k).$$

Hence, unless $\psi(k) \widehat{\bar{\phi}}(k - i0) = 1$ on a set of non-zero measure, we have $g_\psi \equiv 0$ which by (110) gives $\bar{g} \equiv 0$ and therefore $\bar{\mathcal{S}} = L^2(\mathbb{R})$.

Now, as $\bar{\phi} \psi = 0$,

$$\alpha \psi(k) \widehat{\bar{\phi}}(k - i0) = -2\pi i \alpha \psi(k) P_- \bar{\phi} = 2\pi i \alpha \psi(k) P_+ \bar{\phi}.$$

Assuming, that $\psi \in L^\infty$ and $P_+ \phi$ or $P_- \bar{\phi} \in L^\infty$, we see that $\bar{\mathcal{S}}_\alpha = L^2(\mathbb{R})$ for sufficiently small $|\alpha|$, as $|\alpha \psi(k) \widehat{\bar{\phi}}(k - i0)| < 1$ and (111) implies $g_\psi \equiv 0$. \square

Proof of Theorem 8.10. Clearly, ϕ and ψ are both bounded and compactly supported, so (78) is satisfied. We have

$$\widehat{\bar{\phi}}(k + i0) \psi(k) = \left(\int_I \frac{dt}{t - k} \right) \chi_{I'}(k) \left(\int_I \frac{dt}{t - k} \right)^{-1} = \chi_{I'}(k)$$

and by Theorem 8.8, we have $\bar{\mathcal{S}} \neq L^2(\mathbb{R})$ and $\text{def } \mathcal{S} = \infty$. On the other hand,

$$\widehat{\bar{\psi}}(k + i0) \phi(k) = \chi_I(k) \int_{I'} \frac{1}{t - k} \left(\int_I \frac{du}{u - t} \right)^{-1} dt \neq 1 \text{ a.e. on } I,$$

since the inner integral is positive and the outer one negative. Hence, by Theorem 8.8, $\widetilde{\bar{\mathcal{S}}} = L^2(\mathbb{R})$, and the first part of the theorem is proved.

For part (ii) we investigate the relationship between the M -function and the bordered resolvent in this example. As a first step, we investigate jumps of the M -function across the real axis. Recall that

$$(112) \quad M_B(\lambda) = \left(\text{sign}(\Im \lambda) \pi i - \widehat{\psi}(\lambda) \widehat{\bar{\phi}}(\lambda) - B \right)^{-1}$$

and that here we have

$$\widehat{\bar{\phi}}(\lambda) = \int_I \frac{dt}{t - \lambda} \quad \text{and} \quad \widehat{\bar{\psi}}(\lambda) = \int_{I'} \frac{1}{t - \lambda} \left(\int_I \frac{du}{u - t} \right)^{-1} dt.$$

Suppose $k \in \mathbb{R} \setminus (I \cup I')$. Then both $\widehat{\phi}$ and $\widehat{\psi}$ are analytic near k and the jump of M_B is given by the jump coming from the $\text{sign}(\Im \lambda)\pi i$ term.

Now suppose $k \in I$. Then the difference $\widehat{\phi}(k + i\varepsilon) - \widehat{\phi}(k - i\varepsilon)$ can be found using a contour integral and we get

$$\widehat{\phi}(k + i0) - \widehat{\phi}(k - i0) = 2\pi i \chi_I(k).$$

Moreover,

$$\lim_{\varepsilon \rightarrow 0} \widehat{\psi}(k \pm i\varepsilon) = \widehat{\psi}(k) = \int_{I'} \frac{1}{t - k} \left(\int_I \frac{du}{u - t} \right)^{-1} dt < 0$$

is analytic on I . Therefore, the jump of M_B^{-1} at k is $2\pi i(1 - \widehat{\psi}(k))$ and since $\widehat{\psi}(k) < 0$, the M -function will jump at k .

Finally, let $k \in I'$. Then $\widehat{\phi}(\lambda) = \int_I \frac{dt}{t - \lambda}$ is analytic in k , so does not jump, while

$$[\widehat{\psi}(k)] = 2\pi i \left(\int_I \frac{du}{u - k} \right)^{-1}.$$

Therefore, $[\widehat{\phi}(k)\widehat{\psi}(k)] = 2\pi i$. This cancels the jump from the $\text{sign}(\Im \lambda)\pi i$ term, and the formula for the jump of M_B^{-1} in the second part is proved.

We next examine the bordered resolvent. Note that, since $\widetilde{\mathcal{S}} = L^2(\mathbb{R})$, only one projection is necessary here. We first consider the resolvent. Let $(A_B - \lambda)u = v$. Then (noting that $D(\lambda) = 1$ in our situation) we have from (36)

$$(113) \quad u(x) = \frac{v(x)}{x - \lambda} - \frac{\psi(x)}{x - \lambda} \left\langle \frac{v}{t - \lambda}, \phi \right\rangle + c_u \left(\frac{1}{x - \lambda} - \frac{\psi(x)}{x - \lambda} \widehat{\phi}(\lambda) \right).$$

We see immediately from (113) that the non-bordered resolvent (as the sum of a multiplication operator with a jump and finite rank operators) has a non-trivial jump across the whole real axis. Moreover, we get

$$(114) \quad \Gamma_1 u = \widehat{v}(\lambda) - \widehat{\psi}(\lambda) \left[\left\langle \frac{v}{t - \lambda}, \phi \right\rangle + c_u \widehat{\phi}(\lambda) \right] + c_u \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{x - \lambda},$$

where the value of the limit of the integrals in the last term is given by $\pi i \text{sign} \Im(\lambda)$. From (36) we have

$$(115) \quad c_u = M_B(\lambda) \left(-\widehat{v}(\lambda) + \left\langle \frac{v}{t - \lambda}, \phi \right\rangle \widehat{\psi}(\lambda) \right).$$

Since we are only interested in the restricted resolvent, we now fix $v \in \widetilde{\mathcal{S}}$. In this example both $\phi, \psi \in L^2 \cap L^\infty$, so we may apply the final part of Theorem 8.5 to obtain that

$$(116) \quad v|_{I'} = \widehat{v\chi_I}(k + i0) \left(\int_I \frac{dt}{t - x} \right)^{-1} \Big|_{I'},$$

while v is an arbitrary L^2 -function outside I' .

Now consider the jump of $u(x, \lambda) = (A_B - \lambda)^{-1}v(x)$ for $k \in I'$:

$$\begin{aligned}
& u(x, k + i0) - u(x, k - i0) \\
&= \underbrace{v(x)2\pi i\delta(x - k) - \psi(x)2\pi i\delta(x - k)\widehat{v\bar{\phi}}(k + i0)}_{\text{cancel due to (116)}} - \frac{\psi(k)}{x - k}2\pi i v(k) \underbrace{\bar{\phi}(k)}_{=0} \\
&+ c_u(k + i0) \underbrace{\left(2\pi i\delta(x - k) - 2\pi i\delta(x - k)\psi(x)\widehat{\bar{\phi}}(k)\right)}_{=0, \text{ as } \psi(k)\widehat{\bar{\phi}}(k)=1 \text{ in } I'} \\
&+ \left(\frac{1}{x - k} - \frac{\psi(x)}{x - k}\widehat{\bar{\phi}}(k)\right) M_B(k) \underbrace{\left(-2\pi i v(k) + \left\langle \frac{v}{t - k}, \phi \right\rangle 2\pi i \psi(k)\right)}_{=0 \text{ due to (116)}} \\
&= 0 \text{ on } I'.
\end{aligned}$$

Here, we have used that $[M_B(k)] = 0$ for $k \in I'$.

We now consider the jump for $k \in I$, where ψ is zero.

$$\begin{aligned}
(117) \quad u(x, k + i0) - u(x, k - i0) &= v(x)2\pi i\delta(x - k) + \frac{[c_u(k)]}{x - k - i0} \\
&+ c_u(k - i0)2\pi i\delta(x - k).
\end{aligned}$$

Next, we use that for any function f ,

$$[(1/f)(k)] = -\frac{[f(k)]}{f(k + i0)f(k - i0)}$$

to obtain

$$\begin{aligned}
(118) \quad [c_u(k)] &= -[(1/c_u)(k)]c_u(k + i0)c_u(k - i0) \\
&= -[M_B^{-1}] \frac{c_u(k + i0)c_u(k - i0)}{-\widehat{v}(k - i0) + \widehat{\psi}(k - i0)\widehat{v\bar{\phi}}(k - i0)} \\
&\quad - \frac{c_u(k + i0)c_u(k - i0)}{M_B(k + i0)} \left[\frac{1}{-\widehat{v}(\lambda) + \left\langle \frac{v}{t - \lambda}, \phi \right\rangle \widehat{\psi}(\lambda)} \right] (k) \\
&= [M_B^{-1}]M_B(k + i0)M_B(k - i0) \left(\widehat{v}(k + i0) - \widehat{\psi}(k)\widehat{v\bar{\phi}}(k + i0) \right) \\
&\quad + M_B(k - i0) \left(2\pi i v(k) - \widehat{\psi}(k)2\pi i v(k) \right).
\end{aligned}$$

Here, we have used the notation $[F(\lambda)](k)$ for the jump of the function F of λ at the point k .

To show that the restricted resolvent has a jump, it is sufficient to show that there exists $v \in \overline{\mathcal{S}}$ such that the righthand side of (117) does not vanish identically. Since $\frac{1}{x + i0} = \pi i\delta(x) + v.p.(\frac{1}{x})$ and the delta-functions cannot cancel the principal value, it is sufficient to show that there exists $v \in \overline{\mathcal{S}}$ such that $[c_u(k)] \neq 0$. To do this, choose $v = 0$ on $I \cup I'$, but $v \in L^2$ not identically zero and sufficiently smooth. Then $v \in \overline{\mathcal{S}}$ and (118) becomes

$$[c_u(k)] = [M_B^{-1}(k)]M_B(k + i0)M_B(k - i0)\widehat{v}(k + i0) \neq 0,$$

as all terms of the product are non-zero. $M_B(k + i0)$ and $M_B(k - i0)$ are non-zero a.e. in k , as they are boundary values of a non-zero analytic function (see [28]).

Now, let $k \in \mathbb{R} \setminus I \cup I'$. Then

$$\begin{aligned}
u(x, k + i0) - u(x, k - i0) &= 2\pi i v \delta(x - k) - \underbrace{\psi(x) 2\pi i \delta(x - k)}_0 v \hat{\phi}(k + i0) \\
&\quad - \underbrace{\frac{\psi(k)}{x - k}}_0 2\pi i v(k) \bar{\phi}(k) + c_u(k + i0) \left(2\pi i \delta(x - k) - \underbrace{2\pi i \delta(x - k) \frac{\psi(x)}{x - k}}_0 \hat{\phi}(k) \right) \\
&\quad + \left[M_B(k + i0) \left(-2\pi i v(k) \delta(x - k) + \left\langle \frac{v}{x - k}, \phi \right\rangle \underbrace{2\pi i \psi(k) \delta(x - k)}_0 \right) \right. \\
&\quad \left. + [M_b(k)] \left(-\hat{v}(k - i0) + \widehat{v\phi}(k - i0) \hat{\psi}(k - i0) \right) \right] \left(\frac{1}{x - k - i0} - \frac{\psi(x)}{x - k - i0} \hat{\phi}(k - i0) \right) \\
&= 2\pi i v(x) \delta(x - k) + c_u(k + i0) 2\pi i \delta(x - k) - 2\pi i v(x) \delta(x - k) \frac{1}{x - k - i0} M_B(k + i0) \\
&\quad - M_B(k + i0) M_B(k - i0) 2\pi i (1 - \hat{\psi}(k)) (-\hat{v}(k) + \widehat{v\phi}(k) \hat{\psi}(k)).
\end{aligned}$$

Therefore, the most singular term is $-2\pi i v(x) M_B(k + i0) \frac{\delta(x - k)}{x - k - i0}$ which cannot be cancelled. Clearly the coefficient does not vanish for any v supported outside $I \cup I'$ with $k \in \{x \in \mathbb{R} : v(x) \neq 0\}$. \square

Proof of Theorem 8.11. (1) Let

$$(119) \quad F_{\pm} := P_{\pm} \bar{g} \mp \frac{2\pi i}{D_{\pm}} (P_{\pm} \bar{\phi}) P_{\pm} (\psi \bar{g}) = 0.$$

We consider the condition (70). Then $g \in \mathcal{S}^{\perp}$ implies $F_{\pm} \equiv 0$ and $[F] = 0$. On Ω we have

$$\frac{\left\langle \frac{1}{x - \mu}, \phi \right\rangle \left\langle \frac{1}{x - \mu} \psi, g \right\rangle}{1 + \left\langle \frac{\psi}{x - \mu}, \phi \right\rangle}$$

is analytic a.e., so its jump is zero. Therefore

$$2\pi i g(k) = \left\langle \frac{1}{x - \mu}, g \right\rangle|_{\Omega} = 0 \text{ a.e.}$$

and g vanishes a.e. on Ω . Moreover, since our conditions are symmetric in ϕ and ψ , we immediately get that also $\tilde{\mathcal{S}}^{\perp} \subseteq L^2(\Omega^c)$.

(2) Consider F_{\pm} from (119). We need to show that if F vanishes, then so does g . We have

$$D_{\pm}(\mu) = 1 + \alpha \int_I \frac{\bar{\phi}}{x - \mu} dx,$$

so

$$F_{\pm}(\mu) = \left\langle \frac{1}{x - \mu}, g \right\rangle - \frac{\alpha \int_I \frac{\bar{\phi}}{x - \mu} dx \int_I \frac{\bar{g}}{x - \mu} dx}{1 + \alpha \int_I \frac{\bar{\phi}}{x - \mu} dx}.$$

Since $g \in \mathcal{S}^{\perp}$, the first part of the theorem implies $g|_{I^c} = 0$ a.e., so

$$F_{\pm}(\mu) = \int_I \frac{\bar{g}}{x - \mu} dx \frac{1}{1 + \alpha \int_I \frac{\bar{\phi}}{x - \mu} dx}.$$

Clearly, $\left(1 + \alpha \int_I \frac{\bar{\phi}}{x-\mu} dx\right)^{-1} \neq 0$ for a.e. μ , so we have that for all $\mu \notin \mathbb{R}$

$$\int_I \frac{\bar{g}}{x-\mu} dx = \left\langle \frac{1}{x-\mu}, g \right\rangle = 0.$$

As in the proof of Lemma A.1, this implies $g \equiv 0$. □

Proof of Theorem 8.12. From Theorem 8.11, we know that $L^2(\Omega') \subseteq \overline{\mathcal{S}} \cap \widetilde{\mathcal{S}}$. Choose $v, \tilde{v} \in L^2(\Omega')$. By assumption, we know $\langle (A_B - \lambda)^{-1}v, \tilde{v} \rangle$. Noting that $v\phi = 0$ and $\tilde{v}\psi = 0$, we get from (35) and (36) that

$$(120) \quad \langle (A_B - \lambda)^{-1}v, \tilde{v} \rangle - \left\langle \frac{v}{x-\lambda}, \tilde{v} \right\rangle = -M_B(\lambda) \left\langle \frac{v}{x-\lambda}, \mathbf{1} \right\rangle \left\langle \frac{1}{x-\lambda}, \tilde{v} \right\rangle.$$

Choosing $v, \tilde{v} \geq 0$ and not identically zero, we can divide λ -a.e. and obtain

$$(121) \quad M_B(\lambda) = \frac{\left\langle \frac{v}{x-\lambda}, \tilde{v} \right\rangle - \langle (A_B - \lambda)^{-1}v, \tilde{v} \rangle}{\left\langle \frac{v}{x-\lambda}, \mathbf{1} \right\rangle \left\langle \frac{1}{x-\lambda}, \tilde{v} \right\rangle}.$$
□

A.2. Results with $\phi, \psi \in H_2^+$.

Proposition A.5. *Let $\phi, \psi \in H_2^+$. Then*

$$g \in \mathcal{S}^\perp \iff \begin{cases} \text{(I)} & g \in H_2^+, \\ \text{(II)} & \bar{g} = -\frac{2\pi i}{D_-} \bar{\phi} P_-(\psi \bar{g}) \text{ (a.e.)}. \end{cases}$$

Proof. We consider the conditions in (70). As $\bar{\phi} \in H_2^-$, we have $P_+\bar{\phi} = 0$, giving $P_+\bar{g} = 0$, hence $\bar{g} \in H_2^-$ and $g \in H_2^+$. Since $P_-\bar{g} = \bar{g}$ and $P_-\bar{\phi} = \bar{\phi}$, the second condition in (70) becomes (II). □

Proof of Theorem 8.14 (outline). We use the fact that $\overline{\mathcal{S}} = \overline{\mathcal{T}}$ where \mathcal{T} is as defined in (10): the elements of \mathcal{T} are found by solving $(\tilde{A}^* - \mu)u = 0$ and varying μ over the resolvent set of some appropriate operators A_B . We therefore start by solving

$$(\tilde{A}^* - \mu)u = (x - \mu)u - c_u \mathbf{1} + \langle u, \phi \rangle \psi = 0$$

where $\phi, \psi \in H_2^+$. Dividing by $(x - \mu)$ we find that

$$u = \frac{c_u \mathbf{1} - \langle u, \phi \rangle \psi}{x - \mu}.$$

Taking the inner product with ϕ we get

$$D(\mu) \langle u, \phi \rangle - \left\langle \frac{c_u}{x - \mu}, \phi \right\rangle = 0.$$

There are two cases to consider.

(1) $\mu \in \mathbb{C}_+$. This means $\left\langle \frac{1}{x - \mu}, \phi \right\rangle = 0$, and therefore $D(\mu) \langle u, \phi \rangle = 0$. There are two subcases to consider.

(1a) $D(\mu) \neq 0$ which implies $\langle u, \phi \rangle = 0$, giving $u = \frac{\mathbf{1}}{x - \mu}$ up to arbitrary constant multiples.

- (1b) $D(\mu) = 0$ giving $u = \frac{c_u \mathbf{1} - \tilde{c}\psi}{x - \mu}$ for arbitrary values c_u and \tilde{c} . For any boundary condition B , by suitable choice of the two constants we see that μ belongs to the spectrum of A_B . Therefore these functions need not be added to the space $\overline{\mathcal{S}}$. However, functions $\frac{\mathbf{1}}{x - \mu}$ are in $\overline{\mathcal{S}}$ due to being able to approximate them using neighbouring values of μ .
- (2) We take $\mu \in \mathbb{C}_-$. Then

$$\langle u, \phi \rangle D(\mu) = \left\langle \frac{c_u}{x - \mu}, \phi \right\rangle = -2\pi i c_u \bar{\phi}(\mu).$$

There are some subcases to consider.

- (2a) $D(\mu) \neq 0$ which implies $u = c_u \frac{1 + (2\pi i \bar{\phi}(\mu)/D(\mu))\psi}{x - \mu}$ for arbitrary c_u ;
- (2b) $D(\mu) = 0$, $\bar{\phi}(\mu) = 0$ giving by explicit calculation a two dimensional kernel: $u = \frac{c_u \mathbf{1} - \tilde{c}\psi}{x - \mu}$ for arbitrary values c_u and \tilde{c} ;
- (2c) $D(\mu) = 0$, $\bar{\phi}(\mu) \neq 0$ giving $c_u = 0$ and $u = \tilde{c} \frac{\psi}{x - \mu}$ for any \tilde{c} .

In the case (2b) for any boundary condition B , by suitable choice of the two constants we see that μ belongs to the spectrum of A_B . Therefore these functions need not be added to the space $\overline{\mathcal{S}}$. In the case (2c) the function $\frac{\psi}{x - \mu}$ should be included in $\overline{\mathcal{S}}$. There is only one B for which it is an eigenfunction (formally $B = \infty$), but even for this choice of B it can be approximated by elements from neighbouring kernels with $D(\mu) \neq 0$. Note that this means that $\overline{\mathcal{S}}$ is independent of B as it should be by Proposition 2.8. This proves the formula for $\overline{\mathcal{T}} = \overline{\mathcal{S}}$ in the theorem.

We now obtain the expression for the dimension of \mathcal{S}^\perp , in the generic case $M = 0 = M_0$, when $\psi(x) = \sum_{j=1}^n c_j/(x - z_j)$, where the z_j are distinct, lie in \mathbb{C}_- and the c_j are all non-zero. We know that $g \in \mathcal{S}^\perp$ if and only if g satisfies both conditions (I) and (II) in Proposition A.5: Using the fact that $P_- = I - P_+$ the second condition becomes

$$(122) \quad (1 - 2\pi i P_-(\psi \bar{\phi}) + 2\pi i \bar{\phi} \psi) \bar{g} - 2\pi i \bar{\phi} P_+(\psi \bar{g}) = 0.$$

The first bracket on the left gives D_+ and by (I) we know that $\bar{g} \in H_2^-$ and so (122) becomes

$$D_+(x) \bar{g} - 2\pi i \bar{\phi} \sum_{j=1}^N c_j P_+ \left(\frac{1}{x - z_j} \bar{g} \right) = 0, \quad g \in H_2^+, x \in \mathbb{R}$$

in which $D_+(x)$ are the boundary values on the real line of the function $D_+(\mu) = 1 + \int_{\mathbb{R}} \frac{\psi(x) \bar{\phi}(x)}{x - \mu} dx$, $\mu \in \mathbb{C}_+$. Thus by the Residue Theorem,

$$(123) \quad g \in H_2^+, \quad \bar{g}(x) = \frac{2\pi i \bar{\phi}(x)}{D_+(x)} \sum_{j=1}^N \frac{c_j \bar{g}(z_j)}{x - z_j}, \quad x \in \mathbb{R}.$$

Therefore, by unique continuation of the meromorphic function to the lower half plane (see [28]) \bar{g} is given by

$$(124) \quad g \in H_2^+, \quad \bar{g}(\mu) = \frac{2\pi i \bar{\phi}(\mu)}{D_+(\mu)} \sum_{j=1}^N \frac{c_j \bar{g}(z_j)}{\mu - z_j}, \quad \mu \in \mathbb{C}_-,$$

from which it is immediately clear that the space of all such g is at most N -dimensional. Note that the expression on the right hand side of the equality sign in (124) is not clearly an element of

H_2^- ; to deal with this we substitute the particular ψ under consideration into the formula for D_+ and use residue calculations to obtain the following expression for its analytic continuation to \mathbb{C} :

$$(125) \quad D_+(\mu) = 1 - 2\pi i \sum_{j=1}^N \frac{\bar{\phi}(z_j)}{z_j - \mu}, \quad \mu \in \mathbb{C}.$$

If $D_+(\mu)$ has no zeros in $\bar{\mathbb{C}}_-$ and if $\bar{\phi}(z_j) \neq 0$ for all j then we get

$$\bar{g}(\mu) = 2\pi i \bar{\phi}(\mu) \sum_{j=1}^N \frac{c_j \bar{g}(z_j)}{D_+(\mu)(\mu - z_j)}, \quad \mu \in \mathbb{C}_-$$

and the condition that $\lim_{\mu \rightarrow z_j} \bar{g}(\mu) = \bar{g}(z_j)$ gives no additional restrictions, as can be confirmed by a simple explicit calculation. In this case, therefore, the defect of $\bar{\mathcal{S}}$ is N .

Now suppose D_+ has zeros in $\bar{\mathbb{C}}_-$; for simplicity we are assuming that they all lie strictly below the real axis. We let μ_1, \dots, μ_ν be the distinct poles of $\bar{\phi}/D_+$, with orders p_1, \dots, p_ν and set $P = \sum_{j=1}^\nu p_j$. In order to ensure that g given by (124) lies in H_2^+ we need that the conditions

$$(126) \quad \sum_{j=1}^N \frac{c_j}{(\mu_k - z_j)^n} \bar{g}(z_j) = 0, \quad n = 1, \dots, p_k, \quad k = 1, \dots, \nu,$$

all hold - a total of P linear conditions on the numbers $\bar{g}(z_1), \dots, \bar{g}(z_N)$. We now check that this is a full-rank system. Suppose for a contradiction that there is a non-trivial set of constants $\alpha_{i,k}$ such that

$$\sum_{k=1}^\nu \sum_{n=1}^{p_k} \frac{\alpha_{i,k}}{(\mu_k - z_j)^n} = 0, \quad j = 1, \dots, N.$$

Define a rational function by $F(z) = \sum_{k=1}^\nu \sum_{n=1}^{p_k} \frac{\alpha_{i,k}}{(\mu_k - z)^n}$ so that F has zeros at z_1, \dots, z_N . Observe that $Q(z) := F(z) \prod_{k=1}^\nu (\mu_k - z)^{p_k}$ is a polynomial of degree strictly less than $P = \sum_{k=1}^\nu p_k$, having N zeros. Now $D_+(\mu) \rightarrow 1$ as $\Im(\mu) \rightarrow \infty$, so D_+ has the same number of zeros as poles. In particular, D_+ has at least as many poles in \mathbb{C} as it has zeros in \mathbb{C}_- , giving $N \geq P$. Thus Q is a polynomial of degree $< P \leq N$ having N zeros. This means $Q \equiv 0$, so $F \equiv 0$, and the constants $\alpha_{i,k}$ must all be zero. This contradiction shows that the set of linear constraints on the N values $\bar{g}(z_j)$ has full rank P , and so the set of allowable values for $(\bar{g}(z_1), \dots, \bar{g}(z_N))$ has dimension $N - P$.

The degenerated case leading to non-zero M and M_0 can be analysed similarly by considering the local behaviour of $\bar{\phi}/D_+$ around zeroes of $D_+(x)$ on the real axis. \square

Proof of Proposition 8.15. We follow the construction of Theorem 8.14, assuming additionally now that $\bar{\phi}$ has the same form as ψ :

$$\bar{\phi}(x) = \sum_{k=1}^{\tilde{N}} \frac{d_k}{x - w_k}, \quad w_k \in \mathbb{C}_+ \text{ distinct and } d_k \neq 0.$$

We shall construct ψ and ϕ so that the defect number $N - P$ of Theorem 8.14 takes any value between 0 and $N - 1$, while the corresponding defect number $\tilde{N} - \tilde{P}$ for $\tilde{\mathcal{S}}$ takes any value between 0 and $\tilde{N} - 1$, independently of the value of $N - P$.

In addition to the function $D(\mu)$ appearing in the proof of Theorem 8.14 we now have a function \tilde{D} which, following (125), has the form

$$(127) \quad \tilde{D}(\mu) = 1 - 2\pi i \sum_{k=1}^{\tilde{N}} \frac{d_k \bar{\psi}(w_k)}{w_k - \mu}.$$

The expressions (125,127) can be further developed using the explicit formulae for ϕ and ψ to obtain

$$(128) \quad D(\mu) = 1 - 2\pi i \sum_{j=1}^N \frac{c_j}{z_j - \mu} \sum_{k=1}^{\tilde{N}} \frac{\overline{d_k}}{z_j - \overline{w_k}},$$

$$(129) \quad \tilde{D}(\mu) = 1 - 2\pi i \sum_{k=1}^{\tilde{N}} \frac{d_k}{w_k - \mu} \sum_{j=1}^N \frac{\overline{c_j}}{w_k - \overline{z_j}}.$$

We choose the points z_j and w_k so that $z_j - w_k \gg 1$ and $|z_j| \gg |w_k|$ for all j, k . Then

$$D(\mu) \approx 1 - 2\pi i \sum_{j=1}^N \frac{c_j/z_j}{z_j - \mu} \sum_{k=1}^{\tilde{N}} \overline{d_k}; \quad \tilde{D}(\mu) \approx 1 + 2\pi i \sum_{k=1}^{\tilde{N}} \frac{d_k}{w_k - \mu} \sum_{j=1}^N \overline{\frac{c_j}{z_j}}.$$

From these expressions we can find approximations to the zeros μ_j of D and $\tilde{\mu}_k$ of \tilde{D} ,

$$\mu_j \approx z_j - 2\pi i \frac{c_j}{z_j} \sum_{k=1}^{\tilde{N}} \overline{d_k}, \quad \tilde{\mu}_k \approx w_k + 2\pi i d_k \sum_{j=1}^N \overline{\frac{c_j}{z_j}};$$

these expressions can be written in the form

$$\mu_j \approx z_j - 2\pi i \alpha_j \overline{d}, \quad \tilde{\mu}_k \approx w_k + 2\pi i d_k \overline{a}; \quad a = \sum_{j=1}^N \alpha_j, \quad d = \sum_{k=1}^{\tilde{N}} d_k, \quad \alpha_j = c_j/d_j.$$

Inspecting these expressions one may deduce that it is possible to assign independently any value in $\{1, \dots, N\}$ to the number P of μ_j in \mathbb{C}_- and any value in $\{1, \dots, \tilde{N}\}$ to the number \tilde{P} of $\tilde{\mu}_k$ in \mathbb{C}_- . Thus $N - P$ may take any value in $\{0, \dots, N - 1\}$ and $\tilde{N} - \tilde{P}$ may take any value in $\{0, \dots, \tilde{N} - 1\}$. Since N and \tilde{N} are arbitrary, the proof is complete. \square

Proof. (Statements in Example 8.16.) In this example, for $\lambda \in \mathbb{C}^+$ we have by the residue theorem

$$(130) \quad D_+(\lambda) = 1 + \alpha \int \left(\frac{1}{x - z_1} \cdot \frac{1}{x - w_1} \right) \frac{1}{x - \lambda} dx = 1 + \frac{2\pi i \alpha}{(z_1 - w_1)(\lambda - z_1)} = \frac{\lambda_0 - \lambda}{z_1 - \lambda}.$$

Clearly, this formula also gives the meromorphic continuation of D_+ to the lower half plane. We remark that this differs from D_- which is given by

$$(131) \quad D_-(\lambda) = 1 + \frac{2\pi i \alpha}{(z_1 - w_1)(w_1 - \lambda)}.$$

We now calculate the numbers N, P, M, M_0 from Theorem 8.14. ψ has a simple pole at $z_1 \in \mathbb{C}_-$, hence $N = 1$. As ϕ has no zeroes, $M_0 = 0$. The function D_+ has one pole at $z_1 \in \mathbb{C}_-$, $\bar{\phi}$ has a simple pole at $w_1 \in \mathbb{C}_+$. Thus all poles of $\bar{\phi}/D_+$ in $\overline{\mathbb{C}_-}$ stem from zeroes of D_+ . The only zero of this function is at

$$\lambda_0 = z_1 + \frac{2\pi i \alpha}{w_1 - z_1}.$$

Thus, if $\lambda_0 \in \mathbb{C}_+$ then $P = M = 0$; if $\lambda_0 \in \mathbb{C}_-$ then $P = 1, M = 0$; if $\lambda_0 \in \mathbb{R}$ then $P = 0, M = 1$.

We next show the form of \mathcal{S}^\perp and $\bar{\mathcal{S}}$ in the case $\lambda_0 \in \mathbb{C}_+$. Using $\bar{\phi} \in H_2^+$, from (70), we have $\bar{g} \in H_2^-$ and

$$(132) \quad \bar{g} = -\frac{2\pi i}{D_-} \bar{\phi} P_-(\psi \bar{g}).$$

Hence,

$$(133) \quad \left(1 + \frac{2\pi i\alpha}{(z_1 - w_1)(\lambda - w_1)}\right) \bar{g} = -\frac{2\pi i\alpha}{\lambda - w_1} P_- \left(\frac{\bar{g}}{\lambda - z_1} \right) = -\frac{2\pi i\alpha}{\lambda - w_1} \left(\frac{\bar{g} - \bar{g}(z_1)}{\lambda - z_1} \right).$$

Noting that $\bar{g}(z_1)$ is a free parameter, a short calculation shows that

$$\bar{g} = \frac{-2\pi i\bar{g}(z_1)}{(\lambda - w_1)(\lambda - \lambda_0)} \quad \text{or} \quad g(x) = \frac{const}{(x - \bar{w}_1)(x - \bar{\lambda}_0)}.$$

Now, $f \in \bar{\mathcal{S}}$ iff

$$(134) \quad 0 = \int f \bar{g} = const \int f(t) \left(\frac{1}{t - w_1} - \frac{1}{t - \lambda_0} \right).$$

This is equivalent to $(P_+ f)(w_1) = (P_+ f)(\lambda_0)$. □

A.3. Analysis for the case $\bar{\phi}, \psi \in H_2^+$.

Proof of Theorem 8.18. We use the characterisation of \mathcal{S}^\perp given in (70):

$$\begin{aligned} g \in \bar{\mathcal{S}}^\perp &\iff \begin{cases} P_+ \bar{g} - \frac{2\pi i}{D_+} \bar{\phi} P_+ (\psi \bar{g}) = 0, \\ P_- \bar{g} = 0. \end{cases} \\ &\iff \bar{g} \in H_2^+ \text{ and } \bar{g} = \frac{2\pi i}{D_+} \bar{\phi} \psi \bar{g}. \end{aligned}$$

Since $D_+ = 1 + 2\pi i \psi \bar{\phi}$ on \mathbb{R} we have

$$\bar{g} \in H_2^+ \text{ and } (1 + 2\pi i \bar{\phi} \psi) \bar{g} = 2\pi i \bar{\phi} \psi \bar{g},$$

so $\bar{g} = 0$. □

A.4. The general case $\psi, \phi \in L^2$.

Proposition A.6. *The set E_0 defined in (89) is countable.*

Proof of Proposition A.6. Let $\alpha \in E_0 \setminus \{0\}$ and E be the set of positive measure on which $1 + 2\pi i\alpha(P_+(\bar{\phi}\psi) - \psi(P_+(\bar{\phi}))) = 0$. Set $f = 2\pi i(P_+(\bar{\phi}\psi) - \psi(P_+(\bar{\phi})))$. As $1 + \alpha f|_E = 0$ then $f|_E = -1/\alpha$; this can only be true for a countable set of α , as Lemma A.4 shows. □

Proof of Theorem 8.20. Without loss of generality, we assume $\alpha = 1$. Let E be the set of positive measure from (89). For $\phi \in L^{2+\varepsilon}$, choose $h \in L^2(E) \cap L^\infty(E)$, while if $\phi \in L^4$, then choose $h \in L^2(E) \cap L^4(E)$. Now, set

$$(135) \quad \bar{g} = (P_+ \bar{\phi}) \chi_E h - \bar{\phi} P_- (\chi_E h).$$

By our assumptions on h and in (78), we have $g \in L^2$.

We next show that \bar{g} satisfies the right hand side of (72) pointwise. Note that here and in several other places in this proof we use that $P_- P_+ f = 0$. This is justified as our assumptions on h and in (78) guarantee that the functions f we apply this to are in appropriate function classes. We have

$$\begin{aligned} P_- \bar{g} &= P_- ((P_+ \bar{\phi}) \chi_E h - \bar{\phi} P_- (\chi_E h)) \\ &= P_- ((P_+ \bar{\phi}) P_- (\chi_E h) - \bar{\phi} P_- (\chi_E h)) \\ &= P_- ((P_+ \bar{\phi} - \bar{\phi}) P_- (\chi_E h)) \\ (136) \quad &= P_- (-(P_- \bar{\phi}) P_- (\chi_E h)) = -(P_- \bar{\phi}) P_- (\chi_E h). \end{aligned}$$

Multiplying by $2\pi i\psi$ and using that $D_+ - D_- = 2\pi i\psi\bar{\phi}$ on the real axis by the Sohocki-Plemelj Theorem (see [28]), gives

$$\begin{aligned}
 2\pi i\psi P_- \bar{g} &= -2\pi i\psi(P_- \bar{\phi})P_-(\chi_E h) \\
 &= 2\pi i\psi(-\bar{\phi} + (P_+ \bar{\phi}))P_-(\chi_E h) \\
 (137) \quad &= -(D_+ - D_-) + 2\pi i\psi(P_+ \bar{\phi})P_-(\chi_E h).
 \end{aligned}$$

We rewrite the D_- -term as follows.

$$\begin{aligned}
 (138) \quad D_- P_-(\chi_E h) &= P_-(D_- P_-(\chi_E h)) = P_-((D_- - D_+)P_-(\chi_E h)) + P_-(D_+ P_-(\chi_E h)) \\
 &= P_-((D_- - D_+)P_-(\chi_E h)) + P_-(D_+ \chi_E h).
 \end{aligned}$$

Inserting this in (137), and rearranging gives the identity

$$2\pi i\psi P_- \bar{g} - P_-((D_- - D_+)P_-(\chi_E h)) - P_-(D_+ \chi_E h) = (-D_+ + 2\pi i\psi(P_+ \bar{\phi}))P_-(\chi_E h).$$

Multiplying by $\bar{\phi}$ and using that on E we have $D_+ = 2\pi i\psi(P_+ \bar{\phi})$ this gives

$$2\pi i\bar{\phi}(\psi P_- \bar{g} + P_-(\psi \bar{\phi} P_-(\chi_E h)) - P_-(\psi(P_+ \bar{\phi})\chi_E h)) = -(D_+ - 2\pi i\psi(P_+ \bar{\phi}))\bar{\phi}P_-(\chi_E h),$$

which, noting that $(D_+ - 2\pi i\psi(P_+ \bar{\phi}))\chi_E h = 0$, is the equation on the right hand side of (72).

We now need to choose $h \in L^2(E)$ suitably to obtain an infinite dimensional subspace for the corresponding \bar{g} . Choose $E' \subset E$ with $|E'| > 0$ and sufficiently small such that $\Omega_\phi \not\subset E'$ (as E has positive measure and ϕ is not identically zero this is always possible). Consider

$$(139) \quad \bar{g} = (P_+ \bar{\phi})\chi_{E'} h - \bar{\phi}P_-(\chi_{E'} h).$$

By the above arguments, $g \in \mathcal{S}^\perp$. Moreover,

$$\bar{g}|_{(E')^c} = -\chi((E')^c)\bar{\phi}P_-(\chi_{E'} h).$$

As $\chi((E')^c)\bar{\phi} \not\equiv 0$ and $P_-(\chi_{E'} h)$ are the boundary values of an analytic function and therefore non-zero a.e. on \mathbb{R} , we have $\bar{g} \not\equiv 0$ whenever $P_-(\chi_{E'} h) \not\equiv 0$ (see [28]), which gives an infinite dimensional set of such functions. \square

Proof of Theorem 8.22 (outline). This follows easily from (74) in Proposition 8.1 and standard results on compact operators. The compactness of the difference of Hankel operators follows from [38, Corollary 8.5].

Consider the analytic operator-valued function

$$I + (\mathcal{M} - \mu I)^{-1}\mathcal{K}$$

which is a compact perturbation of I . We need to know the values $\mu \in \mathbb{C}$ for which this operator has non-trivial kernel. Each connected component of $\mathbb{C} \setminus \text{essran } \mathcal{M}$ either contains only discrete (countable) spectrum or else lies entirely in the spectrum. However for large μ , $\{0\} = \ker(I + (\mathcal{M} - \mu)^{-1}\mathcal{K})$, so by the Analytic Fredholm Theorem (see [42]), outside some bounded set there is no spectrum of $\mathcal{M} + \mathcal{K}$. \square

APPENDIX B. PROOFS OF THE RESULTS OF SECTION 9

In this appendix, we collect spectral results for an operator $T := \bar{\phi}P_+\psi$ acting on L^2 . These are closely related to the corresponding results for the Toeplitz operator $T_a : H_2^+ \rightarrow H_2^+$ given by $T_a u = P_+ a u = P_+ a P_+ u$ which can be found, e.g. in [11].

Assumptions 9.1 are assumed to hold throughout.

Proposition B.1. *Define the operators T on L^2 and $T_a : H_2^+ \rightarrow H_2^+$ as above. Then*

- (1) $\sigma_p(T) \supseteq \{a(z) | z \in \mathbb{C}_-\}$.
- (2) $\sigma_p(T) = \{\mu \in \mathbb{C} \mid (a - \mu) \text{ is not an outer function in } \mathbb{C}_-\} \cup \{0\}$.

- (3) $\mu \notin \overline{\text{Ran}_{z \in \mathbb{C}_-} a(z)}$ implies $\mu \in \rho(T)$.
- (4) $\sigma(T) = \sigma_p(T) = \overline{\text{Ran}_{z \in \mathbb{C}_-} a(z)}$.
- (5) $\sigma_p(T) \setminus \{0\} = \sigma_p(T_a) \setminus \{0\}$.

Remark B.2. The proofs of (i)-(iv) are very similar to the standard proofs for Toeplitz operators which can be found, e.g. in [11].

Proof. Proof of 1: Take $u(k) = \frac{\bar{\phi}(k)}{k - z_1}$, $z_1 \in \mathbb{C}_-$. Then

$$\begin{aligned} Tu &= \bar{\phi}P_+\left(\frac{\psi\bar{\phi}}{k - z_1}\right) = \bar{\phi}P_+\left(\frac{a(k)}{k - z_1}\right) \\ &= \bar{\phi}P_+\left(\frac{a(k) - a(z_1)}{k - z_1} + \frac{a(z_1)}{k - z_1}\right) = \frac{a(z_1)}{k - z_1}\bar{\phi}(k) = a(z_1)u(k) \end{aligned}$$

since the first term acted on by P_+ is analytic in \mathbb{C}_- and in $L^2(\mathbb{R})$ and the second is in H_2^+ since $z_1 \in \mathbb{C}_-$.

Proof of 2: We first consider $\mu = 0$. Choosing $\bar{g} = \bar{\phi}h$ for $h \in H_2^-$, we get

$$T\bar{g} = \bar{\phi}P_+\psi\bar{\phi}h = 0,$$

since $\psi\bar{\phi}h \in H_2^-$. Hence all functions in $\bar{\phi}H_2^-$ are eigenfunctions to the eigenvalue 0.

Now let $\mu \neq 0$ and assume that $(a - \mu)$ is an outer function in \mathbb{C}_- . We use that if $f \in H_\infty^-$, then f is outer iff $\overline{fH_2^-} = H_2^-$ (Beurling Theorem, see [28]) and that the functions $(k - z_0)^{-1}$ for $z_0 \in \mathbb{C}_+$ span H_2^- . Therefore

$$\bigvee_{z_0 \in \mathbb{C}_+} (a(k) - \mu) \frac{1}{k - z_0} = H_2^-.$$

Now assume there exists $g \in L^2 \setminus \{0\}$ with $T\bar{g} = \mu\bar{g}$ and set $h = \psi\bar{g}$. Then $h \in L^1 \setminus \{0\}$ and $aP_+h = \mu h$, or $(a - \mu)P_+h = \mu P_-h$. Let $z \in \mathbb{C}_+$, then

$$\int_{\mathbb{R}} (a - \mu)P_+h \frac{dk}{k - z} = \mu \int_{\mathbb{R}} P_-h \frac{dk}{k - z} \equiv 0.$$

Therefore, $P_+h \perp \frac{\bar{a} - \bar{\mu}}{k - \bar{z}}$ for all $z \in \mathbb{C}_+$, i.e.

$$P_+h \perp \bigvee_{\bar{z} \in \mathbb{C}_-} \frac{\bar{a} - \bar{\mu}}{k - \bar{z}} = H_2^+.$$

This implies $P_+h = 0$, giving $P_-h = 0$, so $h = 0$ and, as ψ is non-zero a.e. (due to condition (i) in Assumptions 9.1), we have $g = 0$, so μ is not an eigenvalue of T .

Next let $\mu \neq 0$ and assume that $(a - \mu)$ is not an outer function in \mathbb{C}_- . This implies that there exists $h \in H_2^- \setminus \{0\}$ such that $h \perp (a - \mu)H_2^-$. Now

$$h \perp (a - \mu)H_2^- \iff \bar{h} \perp (\bar{a} - \bar{\mu})H_2^+ \iff (a - \mu)\bar{h} \in H_2^-,$$

so $P_+(a - \mu)\bar{h} = 0$ and $P_+a\bar{h} = \mu P_+\bar{h} = \mu\bar{h}$ (as $\bar{h} \in H_2^+$). This implies

$$\underbrace{\bar{\phi}P_+\psi}_{T}\bar{\phi}h = \mu\bar{\phi}h.$$

As $\phi \in L^\infty$, $\bar{\phi}h \in L^2$ and it is not identically zero (as $\phi \not\equiv 0$ and h is non-zero a.e. by the uniqueness theorem, see [28]), so $\mu \in \sigma_p(T)$.

Proof of 3: We first note that $\mu \notin \overline{\text{Ran}_{z \in \mathbb{C}_-} a(z)}$ implies $\mu \neq 0$ and that $\inf_{z \in \mathbb{C}_-} |a(z) - \mu| > 0$. We want to calculate the resolvent of T at μ . Consider $(T - \mu)g = v$. Since $\psi \neq 0$ a.e. and $(a - \mu)|_{\mathbb{R}}$ is invertible we get (all equalities hold a.e.)

$$(140) \quad \begin{aligned} (T - \mu)g = v &\iff \psi \bar{\phi} P_+ \psi g - \mu \psi g = \psi v \iff (a - \mu)P_+(\psi g) - \mu P_-(\psi g) = \psi v \\ &\iff P_+(\psi g) = \frac{\psi}{a - \mu} v + \frac{\mu}{a - \mu} P_-(\psi g). \end{aligned}$$

Note that, as $\frac{\mu}{a - \mu} \in H_{\infty}^-$, the last term lies in H_2^- . Applying P_+ and P_- to (140), we get

$$P_+(\psi g) = P_+ \frac{\psi v}{a - \mu}, \quad 0 = P_- \frac{\psi v}{a - \mu} + \frac{\mu}{a - \mu} P_-(\psi g).$$

Thus,

$$\psi g = P_+(\psi g) + P_-(\psi g) = P_+ \frac{\psi v}{a - \mu} - \frac{a - \mu}{\mu} P_- \frac{\psi v}{a - \mu}$$

and

$$g = \frac{1}{\psi} P_+ \frac{\psi v}{a - \mu} - \frac{a - \mu}{\psi \mu} P_- \frac{\psi v}{a - \mu} = \frac{v}{a - \mu} - \frac{a}{\mu \psi} P_- \frac{\psi v}{a - \mu}.$$

Formally, we have

$$(141) \quad g = (T - \mu)^{-1} v = \frac{v}{a - \mu} - \frac{\bar{\phi}}{\mu} P_- \frac{\psi v}{a - \mu}.$$

Since $\phi, \psi \in L^\infty$, the linear operator defined by the r.h.s. is bounded in $L^2(\mathbb{R})$. We check the formal calculation of the resolvent:

$$\begin{aligned} (T - \mu) \left(\frac{v}{a - \mu} - \frac{\bar{\phi}}{\mu} P_- \frac{\psi v}{a - \mu} \right) &= \bar{\phi} P_+ \frac{\psi v}{a - \mu} - \frac{\mu v}{a - \mu} - \frac{\bar{\phi}}{\mu} \underbrace{P_+ a P_- \frac{\psi v}{a - \mu}}_{=0} + \bar{\phi} P_- \frac{\psi v}{a - \mu} \\ &= \bar{\phi} \frac{\psi v}{a - \mu} - \frac{\mu v}{a - \mu} = v. \end{aligned}$$

Similarly,

$$\frac{(T - \mu)v}{a - \mu} - \frac{\bar{\phi}}{\mu} P_- \frac{\psi(T - \mu)v}{a - \mu} = v,$$

so $T - \mu$ is invertible.

Now, 4 follows from 1 and 3, as

$$\sigma(T) \subseteq \overline{\text{Ran}_{z \in \mathbb{C}_-} a(z)} \subseteq \overline{\sigma_p(T)} \subseteq \sigma(T),$$

so all three sets must coincide.

Proof of 5: We again solve $T\bar{g} = \mu\bar{g}$. As $\psi \neq 0$ a.e., this is equivalent to $aP_+\psi\bar{g} = \mu\psi\bar{g}$. Setting $h = \psi\bar{g}$ this gives $aP_+h = \mu h$.

Note that if $h \in L^2$ and $\mu \neq 0$ with $aP_+h = \mu h$, then $h = \psi \frac{\bar{\phi} P_+ h}{\mu} \in \psi L^2$, so $\bar{g} = h/\psi \in L^2$. Thus $T\bar{g} = \mu\bar{g}$ is equivalent to $aP_+h = \mu h$. This reduces the problem to considering Toeplitz operators:

$$aP_+h = \mu h \iff \begin{cases} (P_+ a P_+) P_+ h &= \mu P_+ h, \\ P_- a P_+ h &= \mu P_- h \end{cases} \iff \begin{cases} (P_+ a P_+) P_+ h &= \mu P_+ h, \\ P_- h &= \frac{1}{\mu} P_- a P_+ h \end{cases}$$

Thus, P_+h determines P_-h uniquely and we only need to consider the first equation in H_2^+ which shows equality of the point spectra of T and T_a away from 0. \square

Example B.3. We illustrate Proposition B.1 4 with an example: We consider a case where we have no non-zero eigenvalues on the boundary of $\sigma_p(T)$, as $(a - \mu)$ is outer. Let $\bar{\phi}(k) = (k + i)^{-1}$ and $\psi(k) = (k + i)(k - i)^{-4}$. Then $\phi, \psi \in L^2 \cap L^\infty$, $\bar{\phi} \in H_2^-$ and $a(z) = (z - i)^{-4} \in H_1^-$, so Assumptions 9.1 are satisfied. To determine $\text{Ran}_{z \in \mathbb{C}_-} a(z)$, we consider $a(t), t \in \mathbb{R}$ and take the inside of the curve.

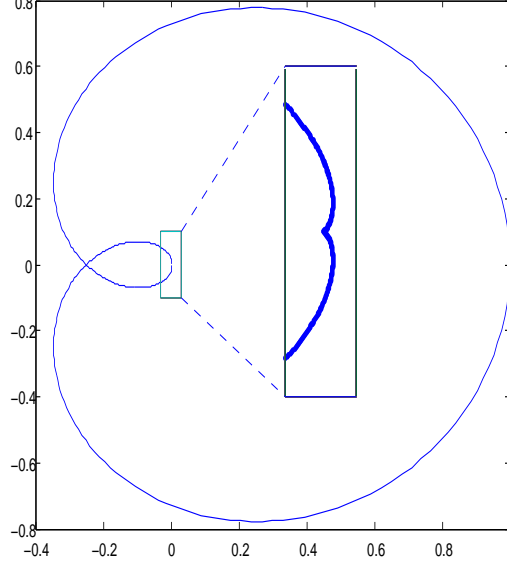


FIGURE 2. The range of $a(t) = (t - i)^{-4}$ for $t \in \mathbb{R}$ with the section around the origin enlarged.

Let

$$x + iy := \frac{1}{(t - i)^4} = \frac{(t + i)^4}{(t^2 + 1)^4} = \frac{t^4 - 6t^2 + 1}{(t^2 + 1)^4} + i \frac{4t(t^2 - 1)}{(t^2 + 1)^4}.$$

We first check that all non-zero points inside the inner curve are in the range. Now, if x is small and negative and $y = 0$, then

$$\frac{1}{(z - i)^4} = x \iff (z - i)^4 = \frac{1}{x} \iff z - i = \frac{1}{\sqrt[4]{-x}} e^{\frac{i(\pi + 2\pi m)}{4}}, \quad m \in \mathbb{Z},$$

so, e.g. for $m = 2$, $z = i + \frac{1}{\sqrt[4]{-x}} e^{\frac{5i\pi}{4}} \in \mathbb{C}_-$, so the point x lies in $\text{Ran}_{z \in \mathbb{C}_-} a(z)$. Similarly, we see that all points between the inner and outer curve lie in $\text{Ran}_{z \in \mathbb{C}_-} a(z)$. Next, we check the points on the inner curve (corresponding to $|t| > 1$): Let $t > 1$ and $(t - i)^{-4} = (z - i)^{-4}$. Then $z - i = (t - i)e^{i\frac{2\pi m}{4}}$, $m \in \mathbb{Z}$. With $m = 3$, $z = i + (t - i)(-i) = -1 + (1 - t)i \in \mathbb{C}_-$.

Therefore, the boundary of the set $\text{Ran}_{z \in \mathbb{C}_-} a(z)$ consists of the outer curve ($|t| \leq 1$) together with the isolated point 0. For these μ , the function $(a - \mu)$ is outer in \mathbb{C}_- . For $\mu = 0$ this is clear. For all other such μ , $(a - \mu)$ takes values outside a cone. This implies that $(a - \mu)^k$ is outer for some sufficiently small positive k which implies that $(a - \mu)$ is outer, since any Herglotz function (i.e. analytic functions on \mathbb{C}_+ with positive imaginary part) is outer.

We finally consider the behaviour at $t = \pm 1$ and $t = \pm\infty$. As $t \rightarrow \pm\infty$, $x \sim t^{-4}$ and $y \sim 4t^{-5}$, so $y \sim 4x^{5/4}$. At $t = 1$, $x \sim \frac{-4-8(t-1)}{16}$, $y \sim \frac{8(t-1)}{16}$, so $y \sim -x - \frac{1}{4}$, and using symmetry of the range of a w.r.t. complex conjugation, we get a cone of angle $\pi/2$ at this point.

Example B.4. The next example shows that in statement 2 of Proposition B.1, it is necessary to add the point $\{0\}$, as it is not always contained in the set $\{\mu : (a - \mu) \text{ is outer}\}$: Let $\alpha_0 \in \mathbb{R}$ and

consider

$$\psi(z) = \frac{(z - \alpha_0)(z + i)}{(z - i)^3} e^{\frac{i}{z}}, \quad \bar{\phi}(z) = \frac{1}{z + i}, \quad \Im z \leq 0.$$

Then $\phi \in H_2^-$,

$$a(z) = \frac{z - \alpha_0}{(z - i)^3} e^{\frac{i}{z}} \in H_1^- \cap H_\infty^-$$

and $0 \notin \text{Ran}_{z \in \mathbb{C}_-} a(z)$. Due to the singular exponential factor, $a = (a - 0)$ is not an outer function.

Proof of Theorem 9.3. Proof of 1: As $a(z) \in H_1^- \setminus \{0\}$, we have $\phi, \psi \neq 0$ a.e. Moreover for $g \in L^2$ we have from (70) and using $\bar{\phi} \in H_\infty^+$

$$(142) \quad g \in \mathcal{S}_\alpha^\perp \iff P_+ \bar{g} = 2\pi i \alpha \bar{\phi} P_+(\psi \bar{g}) \text{ and } P_- \bar{g} = 0 \iff \bar{g} = 2\pi i \alpha \bar{\phi} P_+(\psi \bar{g}).$$

We rewrite this as

$$(143) \quad T\bar{g} = \bar{\phi} P_+(\psi \bar{g}) = \mu_\alpha \bar{g},$$

giving $\mathcal{S}_\alpha^\perp = (\ker(T - \mu_\alpha))^*$.

Next let $g \in \mathcal{S}_\alpha^\perp$ and set $h = \psi \bar{g}$. Then, as $\psi \in L^\infty$ we have $h \in L^2$ and

$$g \in \mathcal{S}_\alpha^\perp \iff \psi \bar{\phi} P_+ h = \mu_\alpha h, h \in L^2 \iff a P_+ h = \mu_\alpha h \iff \begin{cases} P_+ a P_+ h = \mu_\alpha P_+ h, \\ P_- a P_+ h = \mu_\alpha P_- h. \end{cases}$$

For the first equivalence, we note that any L^2 -solution of $\psi \bar{\phi} P_+ h = \mu_\alpha h$ with $\mu_\alpha \neq 0$ is divisible by ψ and $h/\psi \in L^2$.

This shows that $P_+ h$ uniquely determines $P_- h$ via $P_- h = \frac{1}{\mu} (P_- a P_+) P_+ h$ and it is sufficient to consider $P_+ a P_+ h = \mu_\alpha P_+ h$ which gives $\mathcal{S}_\alpha^\perp \subseteq \bar{\psi}^{-1}[(\mu_\alpha P_+ + P_- a P_+) \ker(T_a - \mu_\alpha)]^*$. On the other hand, given $h_+ \in \ker(T_a - \mu_\alpha)$, set $\bar{g} = \bar{\phi} h_+$. Then

$$T\bar{g} = \bar{\phi} P_+(\psi \bar{g}) = \bar{\phi} P_+ a h_+ = \mu_\alpha \bar{\phi} h_+ = \mu_\alpha \bar{g}$$

gives the reverse inclusion, since $\mathcal{S}_\alpha^\perp = (\ker(T - \mu_\alpha))^*$.

Proof of 2: Using the characterisation (143), we need to study the equation

$$(T - \mu_\alpha) \bar{g} = (\bar{\phi} P_+ \psi - \mu_\alpha) \bar{g} = 0.$$

We consider the equation pointwise and multiply by ψ . Setting $h = \psi \bar{g}$, we get $h \in L^2$ and

$$(144) \quad a P_+ h - \mu_\alpha h = 0.$$

By virtue of (144), the fact that a is divisible by ψ and $\mu_\alpha \neq 0$, $h/\psi = \bar{g} \in L^2$. Now, using $h = P_+ h + P_- h$, we find

$$(a - \mu_\alpha) P_+ h = \mu_\alpha P_- h.$$

Thus $(a - \mu_\alpha) P_+ h \perp H_2^+$, or $P_+ h \perp (\bar{a} - \bar{\mu}_\alpha) H_2^+$, which implies $P_+ h \in H_2^+ \ominus (\bar{a} - \bar{\mu}_\alpha) H_2^+$. From (144), this implies

$$h \in \frac{a}{\mu_\alpha} (H_2^+ \ominus (\bar{a} - \bar{\mu}_\alpha) H_2^+)$$

and dividing by ψ (which is non-zero a.e.), we get

$$\bar{g} \in \frac{\bar{\phi}}{\mu_\alpha} (H_2^+ \ominus (\bar{a} - \bar{\mu}_\alpha) H_2^+) = \bar{\phi} (H_2^+ \ominus (\bar{a} - \bar{\mu}_\alpha) H_2^+).$$

Taking complex conjugates and using $(H_2^+)^* = H_2^-$ implies one set inclusion. Conversely, let $\bar{g} \in \bar{\phi} (H_2^+ \ominus (\bar{a} - \bar{\mu}_\alpha) H_2^+)$. Then $\bar{g} = \bar{\phi} f_+$ for some $f_+ \in H_2^+ \ominus (\bar{a} - \bar{\mu}_\alpha) H_2^+$. Then

$$(\bar{\phi} P_+ \psi - \mu_\alpha) \bar{g} = \bar{\phi} P_+ \psi \bar{\phi} f_+ - \mu_\alpha \bar{\phi} f_+ = \bar{\phi} (P_+ a - \mu_\alpha) f_+ = \bar{\phi} P_+ (a - \mu_\alpha) f_+ = 0,$$

as $(a - \mu_\alpha) f_+ \in H_2^-$. Hence $g \in \mathcal{S}_\alpha^\perp$ by part (i).

Proof of 3: Since $(a - \mu_\alpha) \in H_\infty^-$ we have the canonical factorisation

$$a(z) - \mu_\alpha = B_\alpha(z)\Sigma_\alpha(z)G_\alpha(z).$$

In \mathbb{C}_- , $B_\alpha\Sigma_\alpha$ is an inner function and G_α is an outer function. As G_α is outer, by Beurling's Theorem, the closure

$$\overline{(a - \mu_\alpha)H_2^-} = B_\alpha(z)\Sigma_\alpha(z)H_2^-.$$

Thus, by part 2, $\mathcal{S}_\alpha^\perp = \phi(H_2^- \ominus B_\alpha(z)\Sigma_\alpha(z)H_2^-)$. This gives (99), since $\phi \neq 0$ a.e. \square

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